Penn State Graduate Program in Acoustics

Ph.D. Candidacy Questions

Questions are organized in 3 sections:

Vibrations

Acoustics of Fluid Media

Mathematics

VIBRATIONS

1. For a free-free rod of elastic modulus E, length ℓ, cross-sectional area A, and composed of matter with density ρ, there is some resonant frequency for the first longitudinal mode. Suppose a lumped mass M of amount

$$M = \beta \rho A \ell$$

is added to one end of the rod. What should be the value of the constant β so that the lowest resonant frequency is reduced to 75% of its value before the mass is added.

2. A rod of length L is suspended vertically. At the top, it is fixed to a rigid support, and at the bottom, it is attached to a mass-spring system, as shown. The rod has Young's modulus E, density ρ, and cross-sectional area A, while the mass has mass m and the spring has stiffness s. Ignore the effects of gravity.

a) Derive a transcendental equation that could be solved to determine the natural frequencies of the rod.

b) What should the length L of the rod be, in terms of the parameters of the system, if it is desired that the lowest natural frequency be at the frequency

$$\omega_1 = \sqrt{\frac{s}{m}}?$$

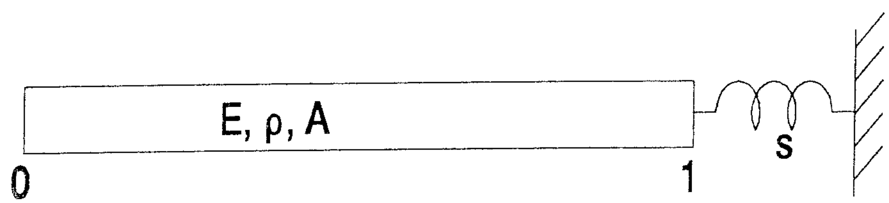
c) For the conditions in part b), determine expressions for the maximum and minimum values of ω_1 which can be obtained by varying the mass and/or stiffness (but keeping L the same as in part b).

3. The first frequency of resonance for a 10-cm string, fixed at both ends, is 100Hz. The density of the string is 1kg/m. What mass must be attached to the center of the string to reduce the frequency of resonance to 90Hz?

Consider an aluminum rod of length 1 m, which is free at one end, and attached to a spring of stiffness s at the other end (see figure). Other parameters of the rod include: Young's modulus (E) = 7.1×10^{10} Pa, density (ρ) = 2700 kg/m^3, and cross-sectional area (A) = 2×10^{-5} m^2.

a) What should the spring constant s be if it is desired that the rod have a natural frequency for longitudinal vibration at 5500 Hz?

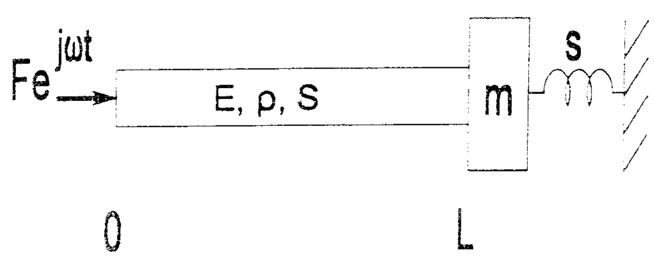
b) In practice, the rod needs to be supported in some fashion. If the rod vibrates at its natural frequency of 5500 Hz, where should two supports for the rod be placed so as to not interfere with the vibration?



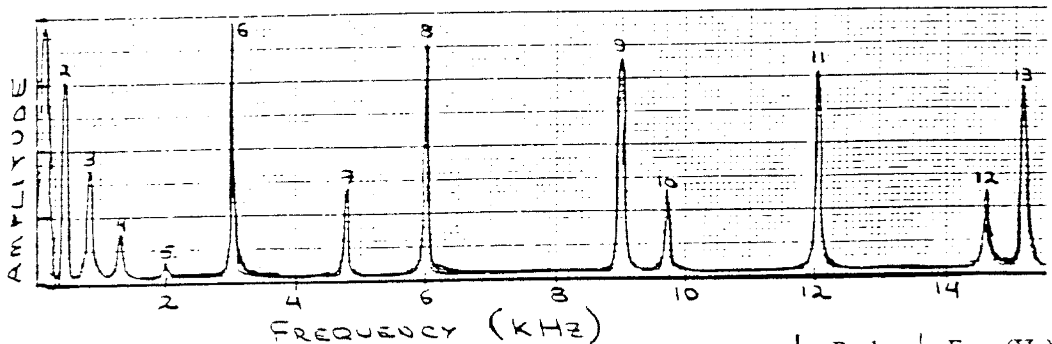
5. A thin rod of length L is fixed at $x = 0$ and free at $x = L$. A static force is applied to the free end of the rod such that the end of the rod is compressed a distance of h. At time $t = 0$, the force is removed and the rod is allowed to vibrate freely. Determine an expression for the displacement of an arbitrary point on the rod for $t \geq 0$.

6. A thin aluminum rod is driven by a time harmonic force, $Fe^{j\omega t}$, at $x = 0$, and it is terminated at $x = L$ by a mass-spring system, as shown below. The mass, spring, and rod are characterized by the following parameters: mass $m = 0.5$ kg, stiffness $s = 5{,}000$ N/m, Young's modulus $E = 7.1 \times 10^{10}$ Pa, density $\rho = 2700$ kg/m^3, cross-sectional area $S = 0.0001$ m^2, and length $L = 1$ m.

a) What is the input impedance at the location of the driving force?
b) What is the approximate fundamental resonance frequency? (Accuracy within 1-2 Hz is sufficient).
c) What would the fundamental resonance frequency be if you removed the mass-spring system from the end of the rod? (i.e. it is a free-end)
d) Explain the results of parts b) and c).



7. **Modes of a Bar.** A bar of circular cross-section has a mass of 301 gm and is 54.0 cm in length and 0.95 cm in diameter. It is mounted in an apparatus which is capable of both exciting and detecting longitudinal, torsional, and flexural resonances. The measurement system leaves both ends in a "free" boundary condition. The resonance spectrum (bar–end velocity amplitude vs. frequency) for the bar is sketched below. The frequencies of the numbered peaks are tabulated at the right. Note that the numbering is based on increasing frequency and has no intentional relation to the modes they identify. Determine Young's and the shear moduli of the bar.



Peak	Freq.(Hz)
1	153
2	422
3	828
4	1,369
5	2,045
6	3,016
7	4,870
8	6,032
9	9,048
10	9,740
11	12,064
12	14,610
13	15,080

8. **Quartz Microbalance.** The mass of extremely thin layers of material can be measured accurately by observing the effect that the layer has on the oscillation frequency of a piezoelectric quartz rod. Determine the frequency shift (in Hz) caused by the deposition of a 2.0 micron (2.0 x 10^{-6} m) layer of gold (ρ_{Au} = 19,300 Kg/m^3) on the right surface of a fixed-free quartz rod (fixed at the left) which is oscillating in its fundamental longitudinal mode at 200 kHz before the gold layer is deposited.

The speed of longitudinal waves in quartz is 5,450 m/sec and the density of quartz is 2,650 Kg/m^3. The cross-sectional area of the quartz rod is not given because it does not affect the result, as long as the diameter of the rod is much less than the wavelength of the excitation.

9. **Elastic moduli.** A sample of syntactic foam is cast into a solid rod of circular cross-section with a length of 14.96 cm and a diameter of 1.55 mm. Its mass is 17.1 gm. The fundamental torsional and longitudinal resonances of the free-free vibrations of the rod were measured to be 4,192 Hz and 6,952 Hz respectively. Determine the shear modulus and the Young's modulus of the syntactic foam.

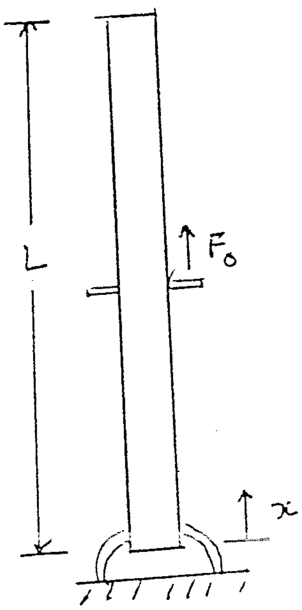
10. Derive an equation for the resonance frequency for longitudinal vibrations for a rod, fixed at one end with a mass attached at the other end. As the mass increases, what will be the range of the first (nonzero) resonance frequency for a steel circular rod with a radius of 1 cm and length 1 meter? [For steel, Young's modulus is 20 x 10^{10} N/m and density is 7800 kg/m^3.]

11. An elastic rod has nominal length L, cross-sectional area A, and is made of material with elastic modulus E and mass ρ per unit volume. The end at $x = 0$ is fixed, that at $x = L$ is free. Before time $t = 0$, a static force F_0 directed parallel to the rod axis is applied at the midpoint ($x = L/2$) of the rod. At time $t = 0$, the force F_0 is suddenly removed.

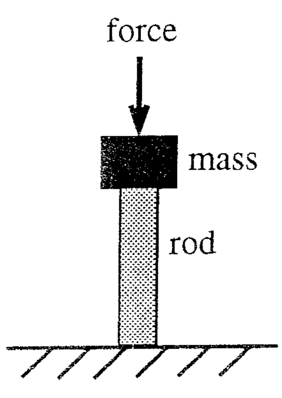
a) At what time t does the end at $x = L$ start to move?

b) What is the peak velocity of the end at $x = 0$?

c) What is the fundamental repetition period of the oscillations following time $t = 0$ at the end $x = L$?



12. A 10 kg mass is attached to the top of a 10 cm long rod ($E=20.\times10^{10}$ N/m^2, and $\rho = 7800$ kg/m^3). The other end of the rod is attached to a rigid foundation. The cross-sectional area of the rod is 1 cm^2. What are the two lowest natural frequencies of the rod/mass for a longitudinal force applied to the mass in the direction along the length of the rod, as shown below?



13. When one makes measurements of the damping in a beam with free ends, it is necessary to support the beam at nodes of bending vibration at resonance to minimize losses of energy into the supports. Suppose one has a beam of length ℓ, cross-sectional area A, mass per unit volume ρ, and bending modulus B. [If the beam is of rectangular cross-section with width w and thickness h, then its cross-sectional area A is wh and its bending moment B is $Ewh^3/12$, where E is the elastic (Young's) modulus.]

a) Derive an transcendental equation that can be solved to determine the lowest natural frequency of such a free-free beam, for which the beam is oscillating in such a manner that it has at least two nodes. Discuss how you would go about solving it and how, in the event more than one solution were possible, you would pick the correct solution. Also, explicitly state how your answer would depend on the parameters ρ, A, B, and ℓ, or on other geometrical and physical properties of the beam.

b) Derive a transcendental equation that can be solved to determine the locations of the beam supports for the case when the beam is oscillating at the frequency determined in part (a).

14. Two flexible structural members each have a rectangular cross-section of thickness h and width w; each has length L and the density of the material that each contains is ρ. Each member is suspended between two supports and pinned at the two ends (simply supported), so that the transverse displacement $y(x,t)$ is 0 at $x = 0$ and at $x = L$. The pins exert no torque on the ends.

(a) Suppose that the first structural member alluded to above is adequately modelled as a Bernoulli beam with no ambient tension, and that the Young's modulus of the material is E_1. Starting from the beam dynamics equation

$$\rho h w \frac{\partial^2 y}{\partial t^2} + \frac{E_1 h^3 w}{12} \frac{\partial^4 y}{\partial x^4} = 0$$

derive expressions for the resonance frequencies of the beam.

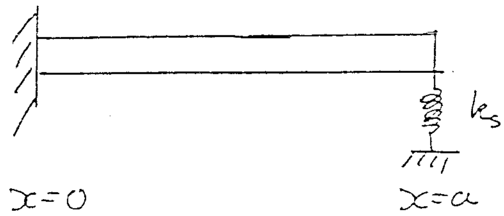
(b) Suppose instead that the second structural member is adequately modeled as a string under tension T, and that the elasticity E_2 of the material is negligibly small. What must the tension T be, in terms of the symbols introduced above, such that the second structural member's lowest resonance frequency will be the same as the lowest resonance frequency of the first structural member?

(c) Given the circumstances described in part (b), suppose each structural member is freely vibrating with the same peak amplitude and at their common lowest natural frequency. What is the ratio,

$$\frac{(\text{Energy})_1}{(\text{Energy})_2}$$

of their total vibrational energies?

15 - Using the following general solution for a beam, derive an expression that can be used to determine the frequencies of resonance for the damped-free beam with a spring at the free end, as shown



The general solution for the displacement of a beam is:

$$w = A \cosh(kx) + B \sinh(kx) + C \cos(kx) + D \sin(kx)$$

The constant for the spring is k_s and k is the free bending wavenumber for the beam. Assume that the spring applies no moments to the beam. Explain how to obtain numerical results for the frequencies of resonance from the expression you derive.

16. Consider a beam of length L which is simply supported at $x = 0$, and terminated with an arbitrary mass at $x = L$. Assume that the mass exerts no moment on the beam.

a) Determine expressions to obtain the natural frequencies and mode shapes for this beam. You do not need to attempt to solve for the natural frequencies.

b) Determine the range of values which the first natural frequency can assume.

Potentially useful information: The following shows the approximate solutions for various transcendental equations.

$\cot(x) = +\tanh(x)$ Solutions: $x \approx \frac{\pi}{4}(1.194, 5.0, 9.0 \ldots)$

$\cot(x) = -\tanh(x)$ Solutions: $x \approx \frac{\pi}{4}(2.988, 7.0, 11.0, \ldots)$

$\tan(x) = +\tanh(x)$ Solutions: $x \approx \frac{\pi}{4}(5.0, 9.0, 13.0, \ldots)$

$\tan(x) = -\tanh(x)$ Solutions: $x \approx \frac{\pi}{4}(3.0112, 7.0, 11.0, \ldots)$

17. The transverse displacement, y, of a thin beam satisfies the Bernoulli-Euler beam equation given by

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} = 0 ,$$

where E is Young's modulus, I is the area moment of inertia, ρ is the density of the beam material, and A is the cross-sectional area of the beam. The beam is simply-supported at $x = 0$, and is driven by a transverse force $Fe^{j\omega t}$ at $x = L$.

a) What is the input impedance of the beam?

b) If a spring is attached at $x = L$ (see figure), what should the spring constant, s, be to increase the lowest resonance frequency by 10 % of its original value (i.e., with no spring)? (Assume the spring exerts no moment.)

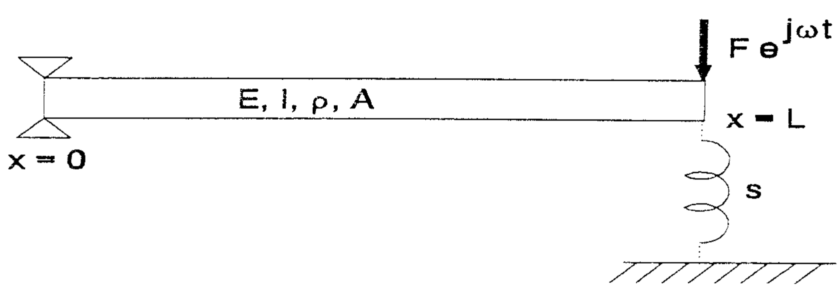
Potentially useful information: The following shows the first three approximate solutions of various transcendental equations.

$\cot(x) = +\tanh(x)$ Solutions: $x \approx \frac{\pi}{4}(1.194,\ 5.0,\ 9.0,\ ...)$

$\cot(x) = -\tanh(x)$ Solutions: $x \approx \frac{\pi}{4}(2.988,\ 7.0,\ 11.0,\ ...)$

$\tan(x) = +\tanh(x)$ Solutions: $x \approx \frac{\pi}{4}(5.0,\ 9.0,\ 13.0,\ ...)$

$\tan(x) = -\tanh(x)$ Solutions: $x \approx \frac{\pi}{4}(3.0112,\ 7.0,\ 11.0,\ ...)$



18. A beam, free at both ends, has a cross-section of 1.0 cm × 1.0 cm and a length of 1.0 m. The lowest resonance frequency for longitudinal waves is measured to be 1.7 kHz.

a) Using the equation for transverse vibration with the beam free at both ends,

$$f_1 = \frac{3.56}{L^2}\left(\frac{Eh^2}{12\,\rho}\right)^{1/2},$$

estimate the frequency of the first bending mode of the free, free beam.

b) What must the thickness of the 1 m beam be for the frequency of the first bending mode to be 60 Hz?

c) What must the length of the 1.0 cm-thick beam be for the frequency of the first bending mode to be 60 Hz?

d) Change either the length or thickness (but not both), to reduce the lowest longitudinal resonance frequency of the free-free beam to 1.0 kHz.

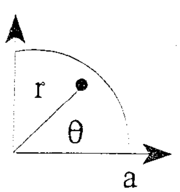
19. A circular membrane fixed at its rim has a radius of 1 meter and a surface density of 0.1 kg/m^3. The membrane is struck in the middle. Ignore air loading effects and determine:

(a) the uniform tension for which the lowest of the excited frequencies is 256 Hz

(b) using the tension derived in part (a), the frequency of the first overtone.

The roots of $J_0(x) = 0$ are 2.405, 5.520, 8.654, ..., and the roots of $J_1(x) = 0$ are 3.83, 7.02, 10.17, ...

20. A flexible membrane fills the first quadrant of a circle of radius a. The membrane is clamped at the boundaries $\theta = 0$, $\theta = \pi/2$, and $r = a$.



a) Find the solution of the vibration equation for this membrane.
b) Find an expression for the eigenfrequencies.
c) What is the lowest eigenfrequency?

21. **Density of modes.** A rigidly-clamped rectangular membrane has a transverse wave speed of 50 cm/sec and its dimensions are 10.0 cm x 14.1 cm.

a. How many modes could be excited by a band-limited noise source which has a frequency range of $70 < f < 80$ Hz?

b. How many modes could be excited by a band-limited noise source which has a frequency range of $145 < f < 155$ Hz?

22. **The Half-Drum.** A previously undiscovered Amazonian tribe was found during a recent expedition. The tribe had developed a unique drum which consisted of a membrane stretched over a semi-circular rigid support as shown below:

(top view)

Determine the tension in the membrane, in N/m, if the lowest mode of vibration occurs at 159 Hz. The diameter is 40 cm and the membrane has a mass density per unit area $\rho_A = 50\ \mathrm{gm/m^2}$. The following table of values for the zeros $J_m(j_{m,n}) = 0$ and extrema points $dJ_m(j'_{m,n})/dx = 0$ of the Bessel functions of integer order, J_m, may be useful in your calculations:

$j_{m,n}$, (zeros)						
$m \downarrow \setminus n \rightarrow$	0	1	2	3	4	5
0	—	2.40	5.52	8.65	11.79	14.39
1	0	3. 83	7.02	10.17	13.32	16.47
2	0	5.14	8.42	11.62	14.80	17.96
3	0	6.38	9.76	13.02	16.22	19.41
4	0	7.59	11.06	14.37	17.62	20.83
5	0	8.77	12.34	15.70	18.98	22.22

$j'_{m,n}$, (extrema points)					
$m \downarrow \setminus n \rightarrow$	1	2	3	4	5
0	0	3.83	7.02	10.17	13.32
1	1.84	5.33	8.54	11.71	14.86
2	3.05	6.71	9.97	13.17	16.35
3	4.20	8.02	11.35	14.59	17.79
4	5.32	9.28	12.68	15.96	19.20
5	6.41	10.52	13.99	17.31	20.58

23. The Simple-Free Donut Shaped Plate. A circular plate has had its center removed to give a donut shaped plate – for example a DVD. The plate has an outer radius a, which is "free." The inner radius is b, which is simply supported (this is a reasonable boundary condition when the disk is being played in a portable DVD player). Come up with the matrix equation that one must examine to find the natural frequencies of this system. DON'T TRY TO SOLVE THE MATRIX EQUATION. We just want to see if you can get it. Please express derivatives compactly to save space, i.e., write $\frac{d^2 J_m(q)}{dq^2}$ as $J_m''(q)$ and similarly.

24. A uniform circular steel disk of undisclosed diameter d and undisclosed thickness h is supported about its circumference by a massless elastic ring in an infinitely large rigid panel. The elastic ring acts as a spring and the panel is rigid and horizontal. When the disk is initially placed in the ring, the ring and disk are observed to uniformly deflect 1 μm under the influence of gravity. The surface of the disk remains flat during this deflection. [Properties of steel: density $\rho = 7700\,\mathrm{kg/m^3}$, Young's modulus $E = 20 \times 10^{10}\,\mathrm{Pa}$, Poisson's ratio $\nu = 0.28$. The acceleration due to gravity is $g = 9.8\,\mathrm{m/s^2}$.]

a) Estimate the value in hertz of the mechanical natural frequency of this ring-disk system.

b) Suppose the disk has a diameter $d = 40$ cm and a thickness $h = 2$ cm. What is the effective stiffness constant of the elastic ring?

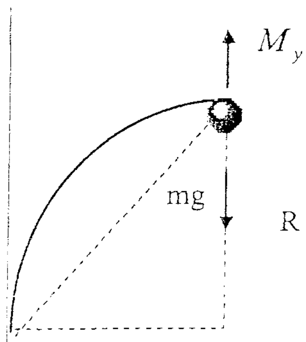
25. A heavy piece of machinery is lowered onto its base using a crane. It was noticed that the springs in the base compressed a total of 2 cm. What is the resonance frequency?

26. A mass M_1 collides inelastically at velocity V_0 with the mass of a mass (M) - spring (K) - damper (R) and starts a transient decaying vibration. Assume the following:

$M_1 = 1$ kg
$V_0 = 1$ m/s
$R = 0.1$ kg/s
$K = 10$ kg/s^2
$M = 2$ kg

a) What is the initial velocity of the system?
b) What is the natural frequency?
c) What is the frequency of resonance?
d) What is the peak displacement?
e) How much time will it take the displacements to decay 99% from the peak?
f) Sketch the displacement, velocity, and applied force to the system assuming the system reaches its initial velocity in 100 ms.

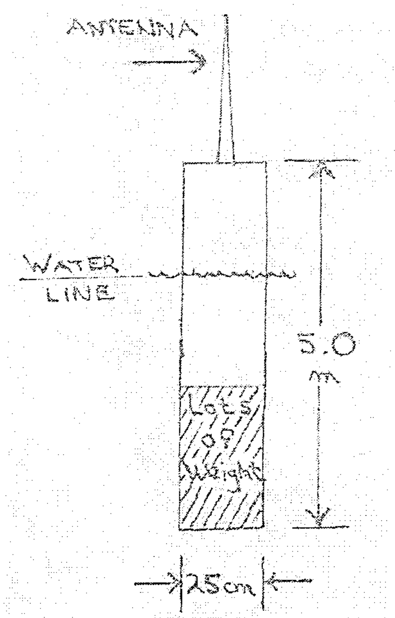
27. A Christmas tree ornament hangs from a branch forming the branch into an arc, where the moment force M_y is balanced by the force of gravity. For small vibrations of the ball and assuming the mass of the branch is negligible, what is the frequency of resonance for the system? Here, Y is Young's modulus, S is cross sectional area of the beam, t is the thickness of the beam, and R is the radius of curvature of the arced beam.



$$M_y = \frac{YS\kappa^2}{R}$$

$$\kappa = \frac{t}{\sqrt{12}}$$

35. **Spar Buoy Oscillations.** An oceanographic instrument package is placed at the bottom of a spar buoy shown at the right. The weight (in air) of the entire buoy (instruments, batteries, tube and antenna) is 150 Kg. The tubular portion (5 m long and 25 cm diameter) is partially submerged as shown in the diagram below.



You may assume that the density of the water, ρ = 1,000 Kg/m^3 and the acceleration due to gravity, g = 9.8 m/sec^2. For this problem, you may neglect the effect of the motion of the water on the effective (moving) mass of the buoy.

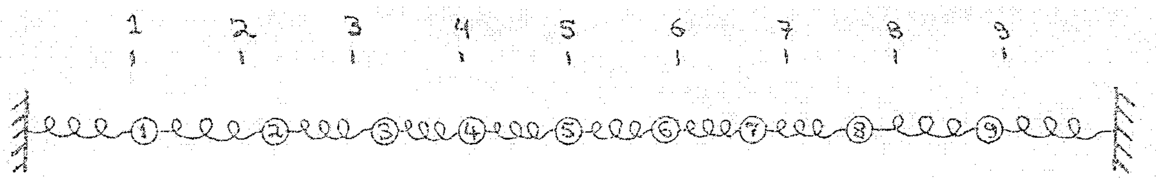
a. How far from the surface of the water is the **top** of the tubular section when it is in equilibrium (at rest)?

b. What is the natural frequency of vibration of the buoy if it is displaced (vertically) from its equilibrium position and released?

c. When the buoy is displaced from equilibrium, the oscillations decay to $1/e$ of their initial value in 30 seconds. Determine the viscous (mechanical) resistance R_m which the water provides to damp the oscillations.

d. Assume that there is a "ring" drawn at the static (calm) water level on the tubular section at the normal equilibrium position which you calculated in part (a) of this problem. If a sinusoidal swell with peak-to-trough amplitude of 1.0 meter passes the buoy position every 20 seconds (*i.e.*, a wave with a 20 second period), what is greatest (peak) distance which the ring will be above or below the instantaneous (moving) air-sea interface? Report the amplitude of the motion of the buoy **relative** to the moving water surface, not with respect to the static (calm) water level.

29. **Coupled oscillators.** Nine objects of equal mass (numbered one through nine) are joined by ten identical springs to two rigid boundaries as shown below.



The masses are constrained to move along the line joining them. In the diagram above, the equilibrium positions of the numbered masses are indicated by the equally spaced, numbered tick marks. The nine masses are shown at an extreme of their collective oscillation corresponding to a normal-mode frequency of 20 Hz. What is the frequency of the fundamental (lowest frequency) mode of oscillation?

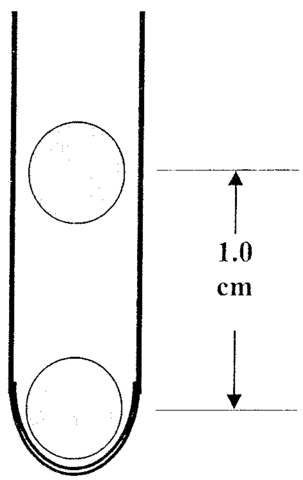
30. A heavy piece of machinery is lowered onto its base using a crane. It was noticed that the springs in the base compressed a total of 2 cm. What is the resonance frequency of this machine on these springs?

31. **Loudspeaker in an enclosure**. A loudspeaker has a free cone resonance frequency of 55.5 Hz. If a US nickel (mass = 5.00 gm) is placed on the speaker's dust cap, the resonance frequency is reduced to 46.3 Hz. After the nickel is removed, the speaker is installed in a sealed enclosure, and the frequency increases to 80.0 Hz. What is the equivalent stiffness (in Newtons/meter) provided by the elasticity of the air contained within the loudspeaker enclosure?

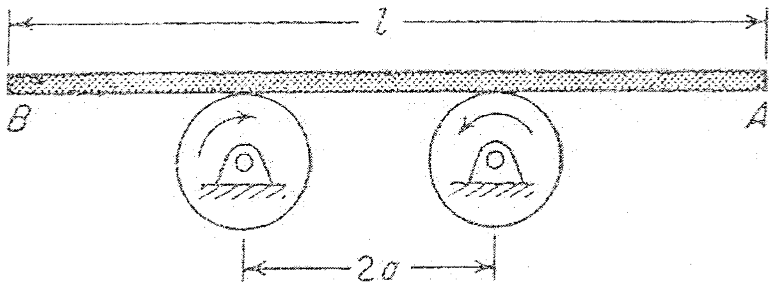
34. **Electrostatic levitation** (15 pts.) Two spheres, made of an electrically insulating material, are contained within an insulating tube as shown below. (You can imagine this as two tiny Styrofoam balls in a drinking straw.) The lower sphere is rigidly attached to the bottom of the tube and the upper sphere is free to move up or down, without significant damping, only in the vertical direction. The mass of each sphere is 0.10 grams. The electrostatic force, F, between the spheres is determined by the charges on the spheres, Q_{upper} and Q_{lower}, the separation between their centers, y, and a constant, $\varepsilon_o = 8.85 \times 10^{-12}$ Farads/meter, known as the Permittivity of Free Space. In this case, the force between the spheres is repulsive, since the charge on both spheres is positive.

$$F = \frac{1}{4\pi\,\varepsilon_o}\,\frac{Q_{upper}\,Q_{lower}}{y^2}$$

At equilibrium, the centers of the spheres are separated by 1.0 cm. What is the frequency of oscillation of the upper sphere if it is displaced from its equilibrium position by a very small amount? (Let the acceleration due to gravity, $g = 9.8$ m/sec^2.)



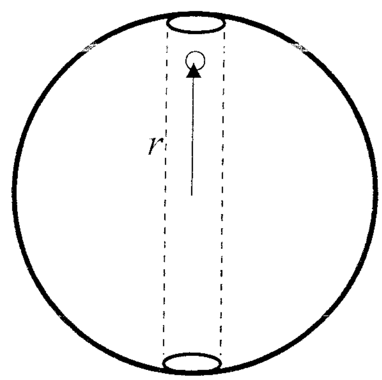
33. **Coefficient of Friction Oscillator.** Two cylindrical rollers are located a distance $2a$ apart; their bearings are anchored and they rotate at an angular speed ω_r in opposite directions, as shown in the figure below. On the top of the rollers rests a bar of length l and weight $W = mg$. Assume a dry coefficient-of-friction of μ between the rollers and the bar. The bar will oscillate back and forth longitudinally executing simple harmonic motion. The restoring force will be the *difference* between the frictional force in the x-direction F_x (along the bar) due to the weight (normal force) of the bar in the y-direction, $F_y = \mu mg$ of the bar on each roller. (Note that if the end of the bar is closer to one roller, the force of the bar on the other roller will be larger and will push the bar back toward the center.)



a. Calculate the frequency ω_o of the bar's simple harmonic motion. Note that the oscillatory frequency ω_o and the rotation frequency ω_r of the rollers are not related. (Hint: It might be conceptually easier to let the length of the bar be twice the roller separation $l = 4a$.)

b. What is the natural frequency of vibration if the direction of both rollers is reversed?

34. **Earth tunnel.** A large diameter solid sphere of uniform density, ρ, is shown at the right. A small diameter cylindrical hole has been bored through the sphere along a diameter. A ball of mass, m, with a diameter slightly smaller than the hole, is dropped from the surface of the sphere down the hole.



At a distance, r, from the center of the sphere, the magnitude of the gravitational potential energy of the ball, $|U(r)|$, is determined by the mass of the sphere, $M(r) = 4\pi\rho r^3/3$, which is included within a radius less than r,

$$|U(r)| = \frac{2\pi \rho G m}{3} r^2$$

Newton's Universal Gravitation Constant, $G = 6.6726 \pm 0.0005 \times 10^{-11}$ m^3/sec^2-Kg. For the planet Earth, $\rho_{Earth} = 5{,}520$ Kg/m^3. If such tunnel could be bored through the Earth, how long would it take for the ball to return to point at the surface where it was initially released? Report your results in minutes. If it were released at a point half way to the center of the earth, how long would it take to return to the half-way release point?

35. A machine mounted on soft resilient mounts behaves like a simple oscillator. Measurements of the impedance of the machine/mount system show that:

(a) As the frequency approaches zero, the impedance has a constant slope with frequency with a value of 3.93 x 10^3 N-s/m at 1 Hz.
the impedance =

(b) At resonance, the impedance is 10 N-s/m.

(c) As the frequency is increased above the resonance frequency, the impedance has a constant slope with frequency with a value of the impedance 1.57 x 10^5 N-s/m at 1 kHz. and the

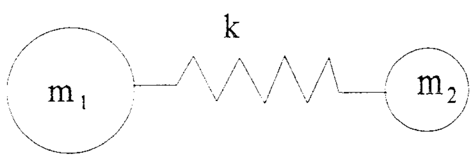
What are the mass, mechanical resistance, stiffness and resonance frequency of the machine/mount system?

36. A tire on an automobile is driven over a sinusoidally periodic series of bumps that are 10 m apart and 0.1 m high. The stiffness of the tire/suspension is 10^4 N/m and the mass of the tire is 10 kg. a.) At what speed will the tire/suspension go into resonance? b.) What is the minimum mechanical resistance required to keep the tire in contact with the surfaces of the bumps?

37. Two masses are attached together by a spring, as sketched below.

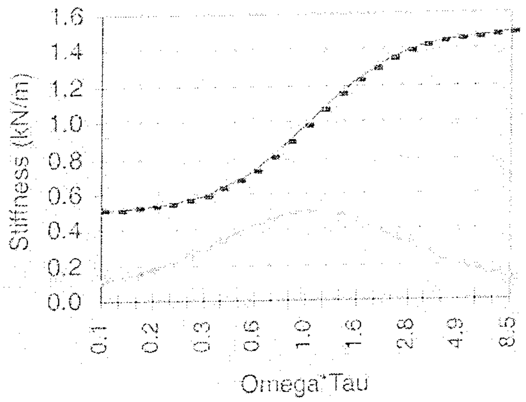
(a) How many degrees of freedom does this system have?

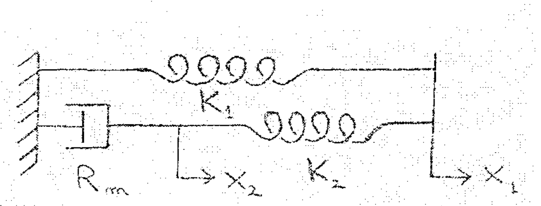
(b) Derive an expression for the frequency, or frequencies, of resonance for all degrees of freedom.



38. A Mass M is attached to a rigid surface through a massless spring with stiffness K and dashpot with damping R. Please derive the following:
a) the resonance frequency
b) the natural (damped response) frequency
c) the frequency of maximum displacement for a sinusoidal applied force
d) the frequency of maximum acceleration for a sinusoidal applied force
e) if the spring has mass M_s, how much of this mass should be included?

39. **Viscoelasticity.** Shown below at the left is a plot (upper line) of the stiffness (in kN/m) as a function of $\omega\tau$ (omega*tau) for a viscoelastic material. To the right of the plot is a model of the material's elastic response that consists of a spring of stiffness k_1 mechanically in parallel with the series combination of a spring of stiffness k_2 and dashpot of mechanical resistance R_m. The lower curve is a plot of the fractional energy dissipated per cycle by the material when the system is strained at a radian frequency ω. That dissipation is maximum for the material at 100 Hz, when $\omega\tau = 1.0$. Determine approximate values ($\pm 10\%$) for k_1, k_2, and R_m that would make the two-spring-damper model at the right represent the viscoelastic behavior shown on the graph at the left.

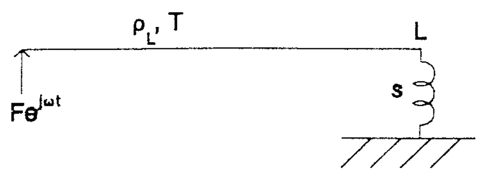




40. A string of length L is driven at x = 0 by a time harmonic force, with a spring of stiffness s attached to the string at x = L, as shown in the figure below. The string has linear density ρ_L and tension T.

a) Determine an expression for the displacement of the string at any arbitrary location along the string

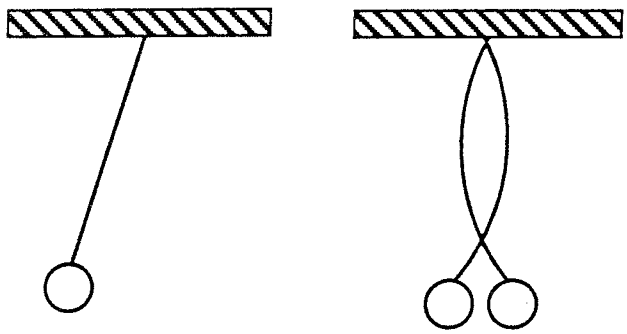
b) With the driving force located at x = 0, as shown, obtain an expression for the mechanical input impedance of the system.



41. Consider a violin string of length 0.5 m and linear density 10 g/m. The bowing force on the string is represented by a square wave with a period of 4 msec. Find the location of the drive (bow location on the string closest to the bridge), frequency of response and tension in the string for the maximum response of the string to be at the first overtone in the excitation of the string.

42. **Guitar String.** The density of steel is 7.7 gm/cm^3 and its yield strength is 200,000 pounds/in^2. What is the highest value of the fundamental frequency of a steel guitar string, rigidly fixed at both ends, which is 80 cm in length?

43. **Pendulum Overtones.** Calculate the ratio of the frequency of the first standing wave mode (one node) to the pendulum mode (no nodes) of vibration for a string of length L=99.3 cm and linear mass density ρ_L = 10.0 gm/m which is fixed rigidly at the top and tensioned at the free end with a 100 gm bob. Let the acceleration due to gravity be g = 9.8 m/s^2.

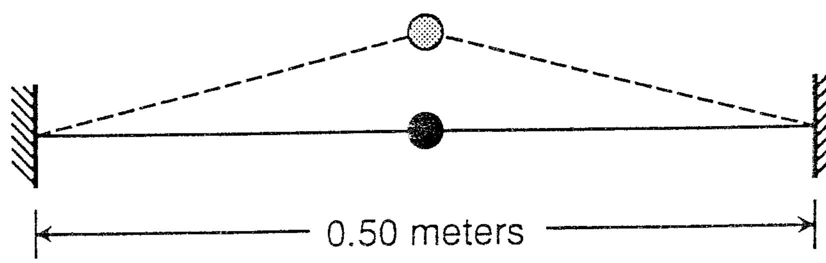


44. Consider a cable of length 1 m and total mass of 1.0 kg that is under a tension of 1000 N and clamped at both ends. For the n = 3 mode, estimate the frequency of resonance for a) the cable before the attachment of a 10 gm accelerometer, b) after the 10 gm accelerometer is attached at 1/2 meter from one end of the cable, and c) after the 10 gm accelerometer is moved to 1/3 meter from one end. Assume that the mode shape does not change and that the only change is in the frequency of resonance. To obtain estimates of these frequencies, it is not necessary to derive a complete solution with the attached masses. Consider the changes in the effective mass of the vibrating cable.

45. **Mass-loaded string.** The mass density of 10 pound test monofilament (plastic) fishing line is 1.35 gm/cm^3 and it is 450 μm in diameter.

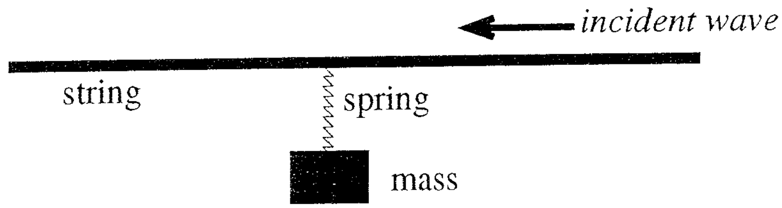
a.) If the tension in the string is 10 lbs, what is the speed of transverse waves on the string?

b.) The fishing line is stretched between two rigid supports which are separated by 50.0 cm and is tensioned as above (10 lbs). At the center of the line is attached a small, spherical lead mass (split shot) of 20 gm. Determine the frequency of the fundamental mode of oscillation of the mass shown schematically below:



c.) Determine the frequency of the second mode of the fixed-fixed string and sketch its displacement. Don't forget to indicate the motion of the centrally located mass.

46. As shown below a mass/spring is attached to an infinite string. The string is under a 100 N tension and has a density of 0.1 kg/m. The mass is 1 kg and the spring constant is $\cancel{10. \times 10^5}$ 1.0×10^5 N/m. A continuous wave is incident at the point of attachment of the string to the spring/mass. Estimate the ratio of the amplitude of the transverse velocity of the transmitted wave to the amplitude of the transverse velocity of the incident wave at 25, 50, and 100 Hz. (The mass/spring is attached at $x = 0$, and the incident wave is propagating in the $-x$ direction.)

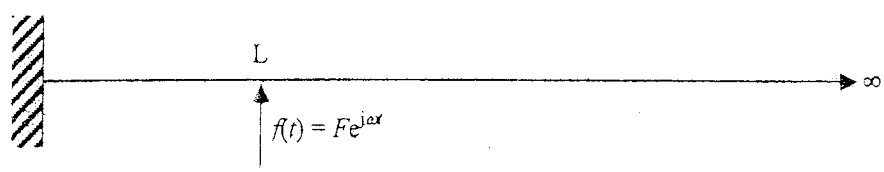


47. **A uniform string under a tension of 15 N has a transverse displacement given by**

$$y(x,t) = 1.73\cos(60t - 12x) \quad ,$$

where the units associated with the numbers are mm, s^{-1}, and m^{-1}, respectively. Calculate the time-average power of the wave.

48. Consider the fixed, infinitely long string shown below. The linear density ρ_L is 0.005 *kg/m* and the wave speed c is 100 *m/s*. It is driven at $L = 1.0\ m$ from the fixed end with a force having an amplitude of 5 *N*.



a) What are the resonance frequencies?
b) When driven at its fundamental resonance frequency, what is the displacement amplitude at $x = L$?
c) How much power is delivered to the string when it is driven at its fundamental resonance frequency?

49. A string of mass/length of 10-g/m is suspended from a rigid point in an elevator shaft 301-m deep. Assume the string hangs straight and is 300-m long.

a) What is the sting wave speed as a function of position?

b) If you pluck the string at the top, would you expect a reflection from the bottom?

50. The equation for the motion of a stiff string is

$$\rho S \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} - ES\kappa^2 \frac{\partial^4 y}{\partial x^4}$$

where ρ is the volume density, S is the cross-sectional area, T is the tension, E is Young's modulus, $\kappa = a/2$ is the radius of gyration and a is the radius. For a 1-meter-long string with $T = 250$ N, $a = 2$ mm, and $\rho = 1000$ kg/m^3, find the value of Young's modulus that will increase the frequency of resonance of the fundamental mode of the string by 10% above the frequency of resonance for the string with no stiffness.

51. A piano string is struck at $x = L/10$ from one end, where L is the length of the string. The velocity of the hammer at the time of contact is U. The mass per unit length of the string is ρ and its tension is T. Determine the frequencies and energies of the response of the string.

52. A string of density 0.01 kg/m is stretched with a tension of 5 N from a rigid support at one end to a device producing transverse periodic vibrations at the other end. The length of the string is 0.5 m, and it is observed that when the driving frequency has a given value, the nodes are spaced 0.1 m apart, and the maximum amplitude of vibration is 0.02 m.

a) Derive an equation for the vibration of the string in response to the driving force.
b) What is the frequency of the driving force?
c) What is the amplitude of the driving force?

53. Consider two waves in a string. The first wave is described by $f_1(x+ct)$ where:

$$f_1(x+ct)/_{t=o} = 2\sin x \quad for\ o \le x \le \pi$$

$$= 0 \text{ otherwise}$$

and the second wave by $f_2(x-ct)$ where

$$f_2(x-ct)/_{t=o} = -\sin x \ for\ -2\pi \le x \le -\pi$$

$$= 0 \text{ otherwise}$$

Sketch the waves at $t = \pi/200$, $\pi/100$ *and* $\pi/50$, *where* $c = 100$.

54. A spring is 2m long and weighs 1 Kg. With no loads applied, the wire loops that make up the spring are well separated. Stretching the spring 10cm produces a tension of 10^{-3} N.

a) One end of the spring is attached to a rigid boundary in a 0g environment. What is the fundamental resonance of the spring in Hz?
b) Now the spring in hung vertically and subjected to gravity (1g). Plot the tension, mass density, and wave speed as a function of position along the spring.
c) Describe physically (in words) how a short wave pulse originating from the free end of the spring propagates and reflects up and down within the spring.

55. A string is fastened between the end walls of a rigid box. The string has length L, is under tension T, and has mass ρ per unit length.

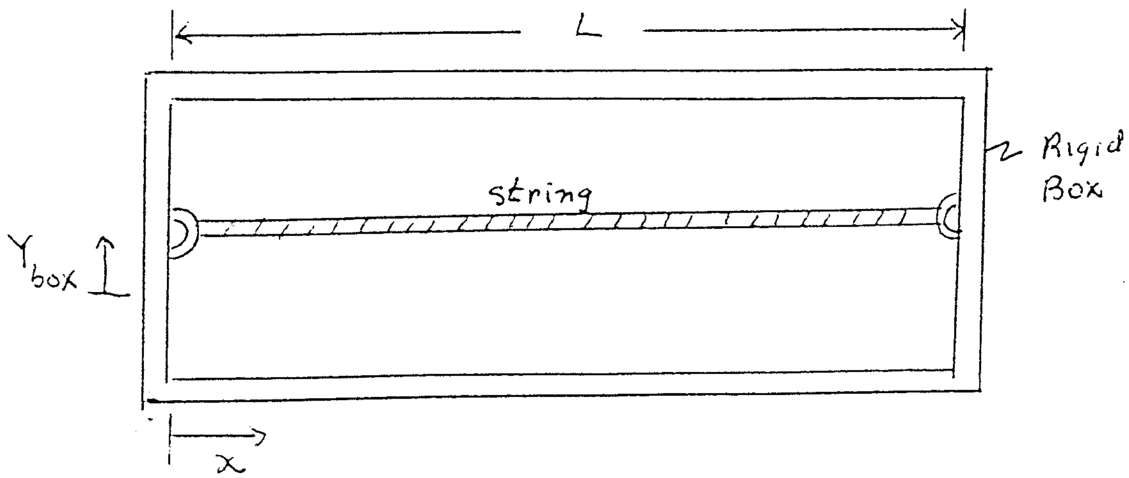
The box is moved up and down with angular frequency ω and amplitude Y_{max}, so that the vertical displacement from the average position is

$$Y_{box}(t) = Y_{max}\cos(\omega t)$$

Let $y(x,t)$ be the vertical displacement of the string at a distance x from its left end, so that $y(o,t) = Y_{max}\cos(\omega t) = y(L,t)$

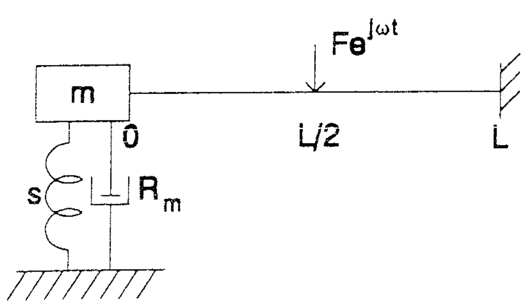
a) Determine the simplest applicable analytical expression for $y(x,t)$.

b) At which ω's (if any) does resonance occur?

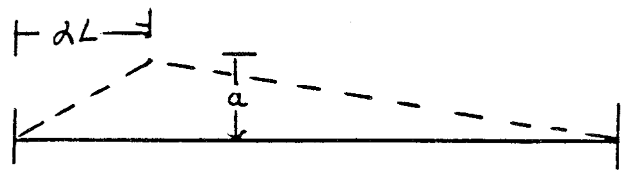


56. A finite string of linear density ρ, length L, and tension T is driven at its midpoint by a time-harmonic excitation force given by $Fe^{j\omega t}$. The string is fixed at $x = L$, and is attached to a mass-spring-damper at $x = 0$ (see figure below).

a) Determine an expression for the mechanical input impedance to the string at the forcing location.
b) For an arbitrary driving frequency, what is the power input to the system by the driving force?
c) If the frequency is such that $kL << 1$, what is the power input to the system?



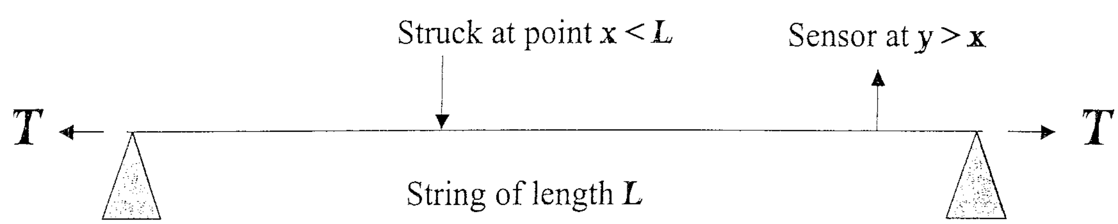
57. Use the wave approach, i.e., solutions of the form $y(x,t) = y_1(ct - x) + y_2(ct + x)$, to sketch the displacement of a plucked string of length L with fixed boundary conditions. Sketch the displacement as a function of x with time as a parameter. Just before the release, the string displacement is shown below.



From your sketches of the string motion, develop a sketch of the transverse force exerted by the string at one of the boundaries of the string. Without doing the mathematics, sketch the spectrum of the force at the end of the string, i.e., the distribution of the harmonics.

58. Consider the following tensioned string of length L that is actually an 1/8-inch diameter ($d = 2a = 3.175$ mm) steel cable. The cable is struck at point x, where $0 < x < L$ and an accelerometer measures the response at some point, y where $x < y < L$. The impulsive excitation results in a string-type transverse wave but also puts a small transient into the tension as well, as to create a bending wave due to shear. Sketch the waveform(s) expected at the sensor point y using the physical parameters listed below.

HINT: Do not try to solve a coupled wave equation – we just want to see what you know about longitudinal, transverse, and bending wave speeds and their effect on the impulse shape.



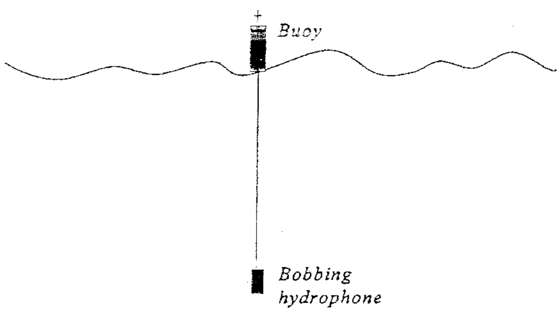
Cable mass/length: $\rho_s = 50$ gr/m
Tension: $T = 667$ N
Young's Modulus: $Y = 19.5 \times 10^{10}$ Pa
Density of steel : $\rho = 7700$ kg/m^3
Bending radius of gyration: $\kappa = a/2 = 0.794$ mm
Bending wave speed:

$c_b = \sqrt{\omega c_l \kappa}$ where the longitudinal wave speed is $c_l = \sqrt{\frac{Y}{\rho}} = 5032.4$ m/s.

Let $L = 300$ m, $x = 100$ m, and $y = 50$ m.

ACOUSTICS OF FLUID MEDIA

/. **Bobbing Hydrophone Response.** Suppose that a pressure hydrophone is suspended some significant distance below the buoy of problem 1. The wave motion is now considered.



Fourier decomposition of the wave vertical motion yields vertical displacement of the hydrophone as a function of surface wave frequency. For example, at 100 Hz the vertical displacement magnitude is 100 μm. Compute the apparent pressure signal (in Pa) picked up by the hydrophone due to this motion at 100 Hz. What is this pressure in dB referenced to 1 μPa?

2. **Schlagwetter-pfeife.** The presence of methane in mines presents a significant hazard due to the possibility of underground explosions. Around 1900, German mines used two whistles to acoustically determine whether there was methane in mines. They pumped the air from the mine out through one whistle and pumped fresh air through the other. Assume the whistles were identical and in good thermal contact (so both surface and sub-surface air passing through the whistles were at the same temperature). Determine the concentration (mole fraction) of methane (CH_4) if the air whistle had a frequency of 440 Hz and the pair of whistles produced 10 beats per second at 20°C. The following data may be useful: $M_{Methane}$= 16.043 a.m.u., $\gamma_{Methane}$ = 1.31, M_{Air} = 28.964 a.m.u., γ_{Air} = 1.403. The Universal Gas Constant, R= 8.3143 J/mole-°K.

3. **Assume that a piston of fixed displacement amplitude and frequency is radiating plane waves into a region filled with an ideal gas. In each of the following cases describe the change in the power output of the piston as the indicated changes are made:**

a) The temperature of the medium remains constant, but the mean pressure is lowered.
b) The mean pressure remains constant, but the temperature is lowered.
c) The medium is originally nitrogen, but is replaced with hydrogen at the same temperature and mean pressure.

4. Let V_0 be the volume occupied by any fixed mass of air with no wave disturbance present. Also, let ρ_0 be the density of air under the same conditions. Now, let there be some small deformation of the air so that V_0 is changed by v, and ρ_0 is changed by ρ. Then the **dilatation** is defined by $\delta = v / V_0$, and the **condensation** by $s = \rho / \rho_0$. These dimensionless ratios describe the instantaneous fractional change in volume and density at a point in a field of sound. They are small quantities. Thus, by assuming that $s\delta << 1$, show that $s \simeq -\delta$.

5. **Sound Speed in an Inert Gas Mixture.** A mixture of helium (M_{He} = 4.003 a.m.u.) and krypton (M_{Kr} = 83.80 a.m.u.) has a sound speed of 456 m/s at 20 °C. What is the molar concentration of the krypton in the gas mixture? The Universal Gas Constant, R = 8.3143 J/mole-K.

6. Using the ideal gas law PV=nRT, where P is the pressure, V is the volume per mole of gas, n is the number of moles, R is the ideal gas constant, and T is the absolute temperature in degrees Kelvin, derive an expression for the speed of sound. Assume PV^γ equals some constant and the equation of state is $\gamma=(C_p/C_v)$ where C_p is the coefficient of specific heat for constant pressure and C_v for constant volume.

(Hints: Assume 1 mole of gas where $V_{mole} = M_{mole}/\rho$, ρ being the density of the gas, use $dV_{mole} = d[M_{mole}/\rho] = -M_{mole}\rho^{-2}d\rho$ to derive $dP/d\rho$ which has the units of velocity-squared.)

a) How does the speed of sound vary for a particular ideal gas with small changes in pressure, density, and absolute temperature?

b) Comparing different ideal gasses at the same static pressure, how does the speed of sound vary from M_{mole}, the gas molar mass, and γ, the ratio of specific heats for the gas?

c) Suppose PV equals a constant rather than PV^γ. Is the acoustic process adiabatic or isothermal?

7. A plane wave in air with 1 pascal peak amplitude at 100 Hz is normally incident in the $+x$ direction on a flat, infinitely massive, rigid surface located at $x = 0$ occupying the y-z plane. In this problem use appropriate ρ_o and c for air, expressing your answers in MKS units. Assume steady state conditions.

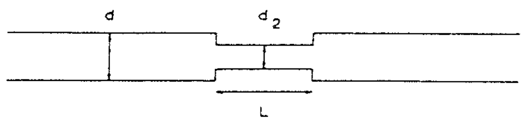
a) What is the normal component of particle velocity next to the surface? What is the acoustic pressure next to the surface? What is the acoustic density deviation next to the surface?

b) Suppose the infinitely massive surface was replaced by a surface with specific acoustic impedance $Z(100\text{ Hz}) = \rho_o c(3. + i20.)$. When the plane wave is again normally incident, determine the extent to which the surface is displaced.

c) Is there a standing wave set up by the direct and reflected waves in part b.)? Justify your answer.

8. Consider an infinitely-long cylindrical duct of diameter d with a short section of length L where the diameter $d_2<d$. The duct has only very low frequency acoustic waves propagating such that $\lambda >> 2d$ and $\lambda >> 2L$.

An Infinite Duct With a Small Constriction of Length L



a) Derive an expression for the pressure transfer function from one side of the short duct section to the other.

b) What parameters of the duct (if any) effect the frequency response of the transmitted waves?

9 The octave-band sound pressure levels measured at a point near a textile loom are as tabulated below:

dB	Center Freq., Hz
67	31.5
72	63
77	125
77	250
82	500
86	1,000
90	2,000
87	4,000
82	8,000
73	16,000

What is going to be the overall sound pressure level (unweighted)? You only need to assume integer-decibel accuracy for your calculations.

10. A noise is generated by 80 pure tones, of different frequencies but identical power. Each pure tone has a sound pressure level of 60 dB (re 20 μPa). Determine the sound pressure level of the total noise.

11. **An Actual e-Mail**

Dear Dr. ------,

I am a project engineer for Moretrench American Corporation in NJ and I ws wondering if
you could help me out with a real life acoutics application. I am bidding a lock and dam
project in Arkansas that requires dewatering of a cofferdam excavation. To do this we have
proposed to use diesel engines to run our pumps. My question,if you want to tackle it is this:

39 diesel engines, approximately 150 ft apart, are installed on the perimeter of the
excavation which measures 1600 ft in diameter. If each engines produces 70dBA @1meter
what is the decibel level in the center of the excavation?

If you can help me out that would be great or if you know anyone who might want to
tackle it please let me know. Thank you for your time and I look forward to hearing from
you soon.

Sincerely,

Your Faithful Professors who ask you to solve this problem using the provided information. State your assumptions, if you use any.

12. Briefly explain why the sound pressure level and the intensity level are approximately numerically equal in air.

13. A rectangular 1.5 × 2 × 2.5m space has rigid walls. An acoustic source that generates white broadband noise in the frequency band from 112 Hz to 135 Hz is placed in the space. What modes of the enclosed space have their natural frequencies in this frequency band, and where should the source be placed in the space to equally excite all of the modes in this frequency band? Assume that the speed of sound is 343 m/s.

14. A plane sound wave is incident at an angle of 45° (or $\pi/4$) on an interface between two fluids. The interface nominally occupies the $y - z$ plane ($x = 0$) and the incident wave direction lies in the $x - y$ plane. The incident side coincides with the region $x < 0$. [Viscosity should be ignored in this problem.] In the fluid on the incident side the sound speed is c_{I} and the ambient density is ρ_{I}. The fluid on the transmitted side has sound speed c_{II} and the ambient density is ρ_{II}.

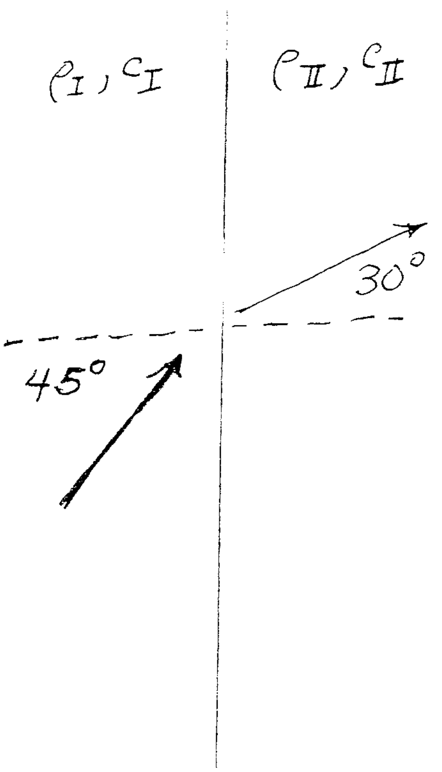
(a) The transmitted wave has a propagation direction that makes an angle of 30° (or $\pi/6$) with the nominal normal to the interface. What is the ratio, $c_{\text{II}}/c_{\text{I}}$?

(b) For the circumstances described above, there is also no reflected wave. What is the ratio, $\rho_{\text{II}}/\rho_{\text{I}}$?

(c) Measurements of the oscillating tangential component of the particle velocity very close to the origin but on the incident wave side ($x = 0^-$) give

$$v_y = V_0 \sin(\omega t)$$

where V_0 is a constant. Given the circumstances of parts (a) and (b), what is the corresponding expression for the tangential component of the particle velocity, again very close to the origin, but in this case on the transmitted wave side ($x = 0^+$)?



15. A sand bottom in seawater is characterized by a density of 1700 kg/m^3 and a sound speed of 1600 m/sec. Consider a plane wave in the seawater which is incident on the planar water/sand interface. For seawater, $\rho_1 c_1 = 1.54 \times 10^6$ Pa-s/m and $c_1 = 1500$ m/sec.

a) What is the critical angle for incidence corresponding to total reflection of the incident wave?

b) For what angle of incidence is the sound power reflection coefficient equal to 0.25?

16. It is desired to use a Helmholtz resonator with a circular opening to attenuate acoustic propagation in a rectangular duct at 100 Hz. The rectangular duct has cross-sectional dimensions of 0.3 × 0.3m, and the space immediately adjecent to the duct that is available for the Helmholtz resonator is 0.3 × 0.3 × 0.5m. Here, the 0.3 × 0.3 m dimensions lie in the plane of the duct wall, and the 0.5 m dimension is the distance normal to the duct wall. Derive a set of design parameters for the resonator under the condition that the circular opening to the resonator has a diameter of 0.1 m. In your answer, define both design parameters that you had to assume, due to constraints (e.g. limitations on available space), and design parameters that you calculated.

Useful information: A short tube (relative to λ) has a mass-like acoustic impedance given by $M_A = \rho L/S$, where L is the effective length of the tube and S is the cross-sectional area. A small cavity (relative to λ) has a stiffness-like acoustic impedance given by $s_A = \rho c^2/V$, where V is the volume of the cavity.

17. You are alone in an acoustics laboratory with rather modest equipment. Your job is to measure the temperature of the air. (Perhaps because of DoD funding cutbacks, you cannot afford a thermometer.) The laboratory equipment you have access to includes: a sharpened pencil, paper, a stopwatch, a large washtub filled with water, several gallons of motor oil, several rubber balloons, a long 5 cm diameter glass tube with open ends, scissors, a 440 Hz tuning fork, a screwdriver, and a measuring tape. All of this may or may not be useful to you. Assume that all of the laboratory equipment is at the same temperature as the air.

a) Give a written experimental procedure for measuring the temperature of the air.
b) Make some estimates as to the quantities you would be measuring and then give a sample calculation. Your calculation procedure can be reasonably crude. i.e. no ACS 515 material need be applied.

18. An acoustic field inside a rectangularly shaped cavity is described by

$$p = 12\cos(3\pi x)\sin(6\pi y)\, e^{-i\omega_1 t}$$

in Pa. The cavity and field are infinite in the z direction. The cavity extends from 0 to 3 m in the x direction and 0 to 2 m in the y direction.

a) Decompose this pressure field into a superposition of plane waves. Give the magnitudes and phases of all the plane waves existing inside the cavity, assuming (and relative to) the $e^{-i\omega_1 t}$ time convention.

b) For this field to exist inside the cavity, the cavity must have boundaries with certain properties. Mathematically and physically describe the boundaries at $x = 0$, $x = 3$, $y = 0$, and $y = 2$.

19. **Spherical Reflection off a Sea Wall.** A vertical rigid concrete wall is adjacent to a deep sea. During a time when the sea is completely calm, a physically tiny spherical sound source is placed 20 m from the sea wall and 20 m below the ocean surface. A hydrophone is also placed 10 m from the sea wall and 10 m below the ocean surface. The hydrophone and sound source are in the same plane perpendicular to both the rigid sea wall and ocean surface. The tiny spherical sound source sends out a brief pulse of sound, and the hydrophone registers the *direct sound* from the source as $p = 0.1\sin(2000\pi t)$ Pa between $t = 0$ and $t = 0.001$ and $p = 0$ Pa otherwise. What will be the reflected sound field(s) the hydrophone will register? You may assume the sea is completely calm and homogeneous with a speed of sound of 1500 m/s and ambient density of 1000 kg/m^3. Please disregard near field effects from the tiny spherical sound source and reflections from this source. Assume the hydrophone is omnidirectional.

20. **Two Fluid Problem: Air and Ethyl Alcohol.** An unbounded halfspace of air (ρ_0=1.21 kg/m^3, c=343 m/s) is over an unbounded halfspace of ethyl alcohol (ρ_0=790 kg/m^3, c=1150 m/s). Assume a 100 Hz continuous sinusoidal plane wave of amplitude 1 Pa is incident from the air onto the air-ethyl alcohol interface at an angle of 45°. At a depth of 1 m below the interface, what is the amplitude of the acoustic pressure in Pa?

21. A plane wave in a lossless fluid is incident on a flat, hard floor at an angle of 45 degrees and reflects without any losses. Let the incident plane wave take the form

$$p(x,z,t) = \mathrm{Re}\left(P_0 e^{i\bar{k}\cdot\bar{x}} e^{-i\omega t}\right)$$

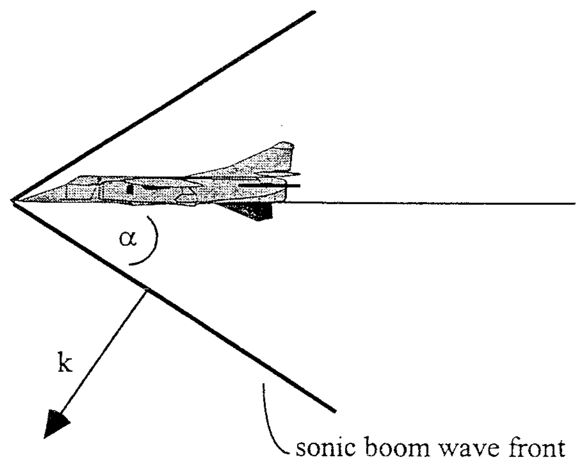
where

$$\bar{k} = \frac{\sqrt{2}}{2}\bar{e}_x - \frac{\sqrt{2}}{2}\bar{e}_z \qquad\qquad \bar{x} = x\bar{e}_x + z\bar{e}_z$$

and this wave is incident from above onto the hard floor located at $z = 0$.

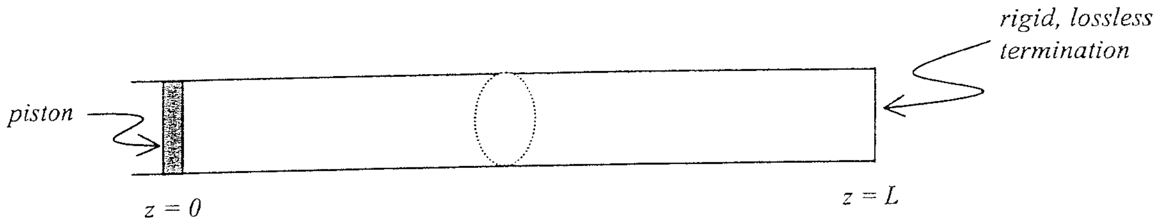
Calculate the position of any nodal points along the z axis (i.e., $x = 0$ for non-negative z) that one may measure with a microphone for this scenerio. There may be many of them. Assume that P_0 is a constant real number. Further, at any one of the nodal points, determine the time averaged acoustic energy density.

22. A supersonic jet is flying parallel to a flat ocean surface. The sonic boom wave front generated by the jet makes an angle α with respect to the flight path. (Refer to the sketch.) The angle α is related to the Mach number M of the aircraft according to $\sin \alpha = 1/M$, where $M = v/c$. v is the speed of the aircraft and c is the speed of sound in the air (assume 20 °C). How fast must the jet be flying for the sonic boom to be transmitted into the ocean? Express your answer in terms of Mach number. The characteristic impedances of air and seawater are 415 Pa s/m and 1.54×10^6 Pa s/m, respectively.



23. Let V_o be the volume occupied by any fixed mass of air with no wave disturbance present. Also, let ρ_o be the density of air under the same conditions. Now, let there be some small deformation of the air so that V_o is changed by v, and ρ_o is changed by ρ. Then the **dilatation** is defined by $\delta = v/V_o$, and the **condensation** by $s = \rho/\rho_o$. These dimensionless ratios describe the instantaneous fractional change in volume and density at a point in a field of sound. They are small quantities. Thus, by assuming that $s\delta << 1$, show that $s \cong - \delta$.

24. Consider plane waves driven by a piston situated at one end of a rigid-walled circular pipe of radius R as shown in the diagram below. The piston oscillates with a speed given by $u(0,t) = U_o e^{j\omega t}$. The termination at $z = L$ is rigid and lossless.



a) Show that the acoustic pressure in the pipe can be expressed as

$$p(z,t) = P_o \cos\left[\hat{k}(L-z)\right]e^{j\omega t}.$$

$\hat{k}$ is the complex propagation constant given by $\hat{k} = k - j\alpha$, where k is the wavenumber and α is the absorption coefficient.

b) Find an expression for P_o in terms of U_o.

c) Show that the time-averaged, active acoustic intensity at $z = 0$ is given by

$$I = \frac{1}{4}\frac{P_o^2}{\rho_o c}\sinh(2\alpha L)$$

when the pipe is driven at resonance. For simplicity, assume that the resonance condition is the same as that for a lossless pipe.

d) What is (are) the dominant attenuation mechanism(s) for an air filled pipe of radius 2.5 cm driven at frequencies in the 200 Hz range?

25. The *sound energy density* is defined as

$$e = \frac{1}{2}\rho_0 u^2 + p^2 / 2\rho_0 c^2$$

where ρ_0 is fluid density, c is sound speed, p is acoustic pressure, and u is acoustic particle velocity.

a) In your own words, describe what the two terms of this equation mean, physically.

b) Derive an expression for e for a single-frequency (ω), pure, plane progressive acoustic wave that is propagating in the x-direction.

c) Prove that the instantaneous acoustic intensity, $I(x,t) = p(x,t)u(x,t)$, is ce.

d) Make a sketch of the amplitudes of p, u, e, and I versus x showing at least one period.

7. Consider a rigid, air-filled pipe of length L that is forced at one end and has a rigid termination at the other:

a) Derive an expression for the input mechanical impedance.

b) Show for $kL << 1$ that the gas inside the pipe acts as a linear spring, where k is the acoustic wavenumber.

26. Define each of the following types of impedance. For each, give an example of how it is used.

a) specific acoustic impedance
b) acoustic impedance
c) characteristic impedance
d) mechanical impedance
e) radiation impedance.

27. Consider a 5-cm diameter water-filled cylindrical waveguide filled with water.

a) Calculate the cutoff frequency for the three lowest-frequency modes.
b) Assume that the waveguide is driven sinusoidally with a plane, rigid piston located at one end of the waveguide. The piston completely fills the cross section of the waveguide. The frequency at which the piston oscillates is between the cutoff frequencies of the second and third lowest-frequency modes as calculated in part (a). Describe the sound field in the waveguide.
c) Suppose now that the piston from part (b) does not fill the cross section of the waveguide. Assume, instead, that the piston has a diameter less than that of the waveguide and that it is mounted so that the center of the piston does not coincide with the axis of the waveguide. The frequency, however, is still the same as that from part (b). Describe the sound field in the waveguide.

28. A pipe of length L and diameter D is driven by a piston at one end and open at the other. The open end is mounted in an infinite rigid baffle. The frequency range of interest is from 100 - 5000 Hz. L is on the same order of magnitude as the acoustic wavelength λ, and $D << L$.

a) Briefly describe the loss mechanisms relevant to this problem.

b) Suppose that the piston delivers 0.1 W of acoustic power to the pipe. Neglecting losses in the pipe, how much power is radiated out of the open end?

c) Given the equation for the radiation impedance of a baffled circular piston, find an expression for the volume velocity at the open end.

d) Explain how you would find the volume velocity of the piston.

29. In the future, water will remain a very important supply for spaceships and space stations. One model would include water being held in rectangular mylar containers. Suppose such a container of water is being transferred between spaceships in vacuum. The container is 0.5 m by 0.5 m by 1.5 m in size. Assuming that mylar is acoustically transparent, what are the first modes of oscillation of the water in a container? Give an expression for the modes in general, and then the eigenfunctions and eigenfrequencies for the lowest three modes. You can also assume that the water containers will be out in the vacuum for such a short time that the water will not freeze.

30. **Atmospheric Sound Sources.** For this problem assume air on a cold day: $\rho = 1.21$ kg/m^3 and $c = 335$ m/s. A siren outdoors generates a nearly sinusoidal warbling sound with SPL 120 dB re 20 μPa at a distance of 10 m. Because of the ground surface the sound is radiated nearly hemi-spherically. The average frequency is 300 Hz. There is no wind, and nonlinear effects and ground impedance effects are negligible.

(a.) How much sound power is radiated?
(b.) Neglecting atmospheric absorption what is the SPL at a distance of 500 m?
(c.) Neglecting atmospheric absorption what is the time average active acoustic intensity at a distance of 500 m?
(d.) Given that atmospheric absorption is approximately $\alpha = 1.1*10^{-3}$ dB/m at 300 Hz, calculate revised estimates for parts (b.) and (c.).

31. An acoustic experiment is performed in an aquarium partially filled with water (c = 1500 m/s and ρ_0 = 1000 kg/m^3). At the bottom of the aquarium an ultrasonic transducer shoots a narrow beam of sound up to the surface of the water at an angle of 45 degrees, and the sound beam reflects back down to the bottom of the aquarium where it hits a distance along the bottom 16 cm from the transducer. See Fig. 1. You may assume the bottom of the aquarium is perfectly absorbing, so the beam does not bounce back up again. You may also assume that the problem is completely 2-dimensional. After making the required measurement the experimenter comes back from lunch and finds that a layer of ethyl alcohol (c = 1150 m/s and ρ_0 = 790 kg/m^3) has carefully been poured on top of the water in the aquarium. (Good grief!) See Fig. 2. Now a second beam of sound comes down and hits the aquarium bottom at a distance of 22 cm from the ultrasonic transducer. Question: how thick is the layer of ethyl alcohol on top of the water assuming that the two liquids do not mix?

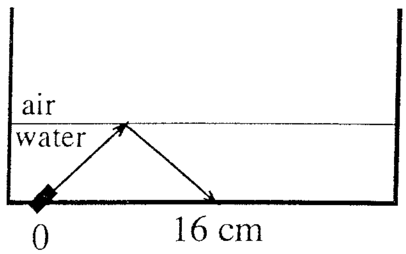


Figure 1.

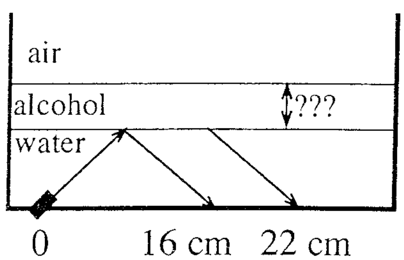
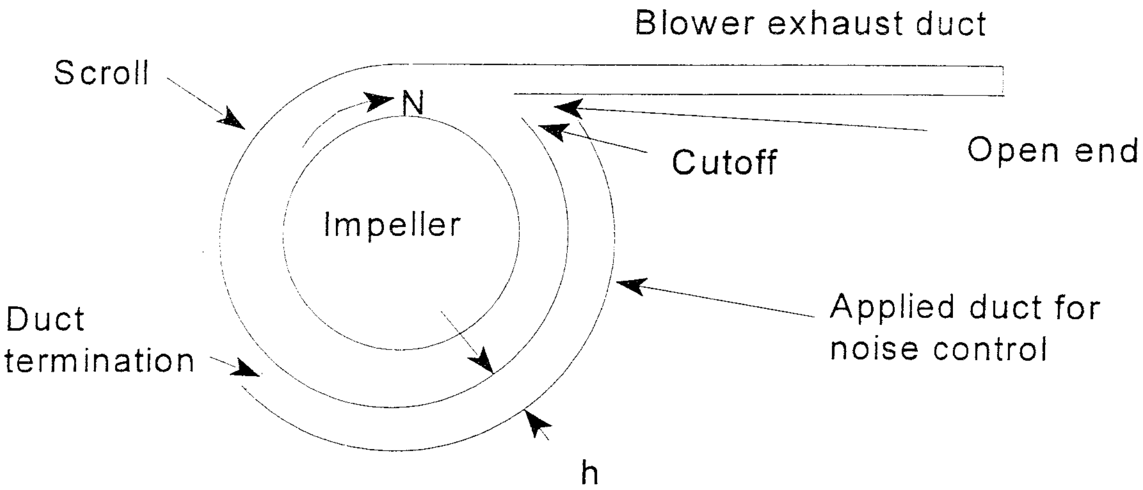


Figure 2

32. A proposal is made to devise a passive noise cancellation system for the noise created at the blade passage tone generated by a centrifugal blower. Sound at the blade passage tone is caused by the local pressure disturbance created when each impeller blade "passes by" the fixed tongue, or cutoff of the scroll. The impeller in this case has 26 blades, and the geometry is shown below.



The applied duct of height, h, has width, W into the plane of the drawing. This duct has an open end near the cutoff and is terminated at arc length, L by some impedance. Sound created at the cutoff propagates in this duct, reflects off of the termination, and returns to the source. The idea is to adjust the dimensions of the applied duct so that destructive interference of the sound energy at the blade pass frequency occurs at the source (the cutoff). Solve the following problems, assuming $L = 0.6$ m, $h = 0.03$ m, and $W = 0.3$ m, and the medium is air under standard conditions:

a) At what blower speed, N (in either in Hz, or rpm's) will destructive interference of the acoustic pressure generated at the blade passage tone take place if the termination is rigid?
b) At what blower speed will destructive interference take place if the termination were simply left open?

33. The reflection losses from an ocean bottom start at 6 dB at normal incidence and decrease to 0 dB at and above 30 degrees off normal. What is the volume density of the bottom?

34. A plane acoustic wave is incident on the ocean floor at an angle of 55°. The ocean floor is quartz sand having a density of 2070 kg/m^3 and a speed of sound of 1730 m/s. The density and sound speed of seawater are 1026 kg/m^3 and 1500 m/s, respectively.

(a) Calculate the angle of transmission.

(b) If the incident wave has a peak pressure amplitude of 100 Pa, what is the SPL (re 1 μPa) of the reflected wave?

(c) What is the SPL (re 1 μPa) of the transmitted wave?

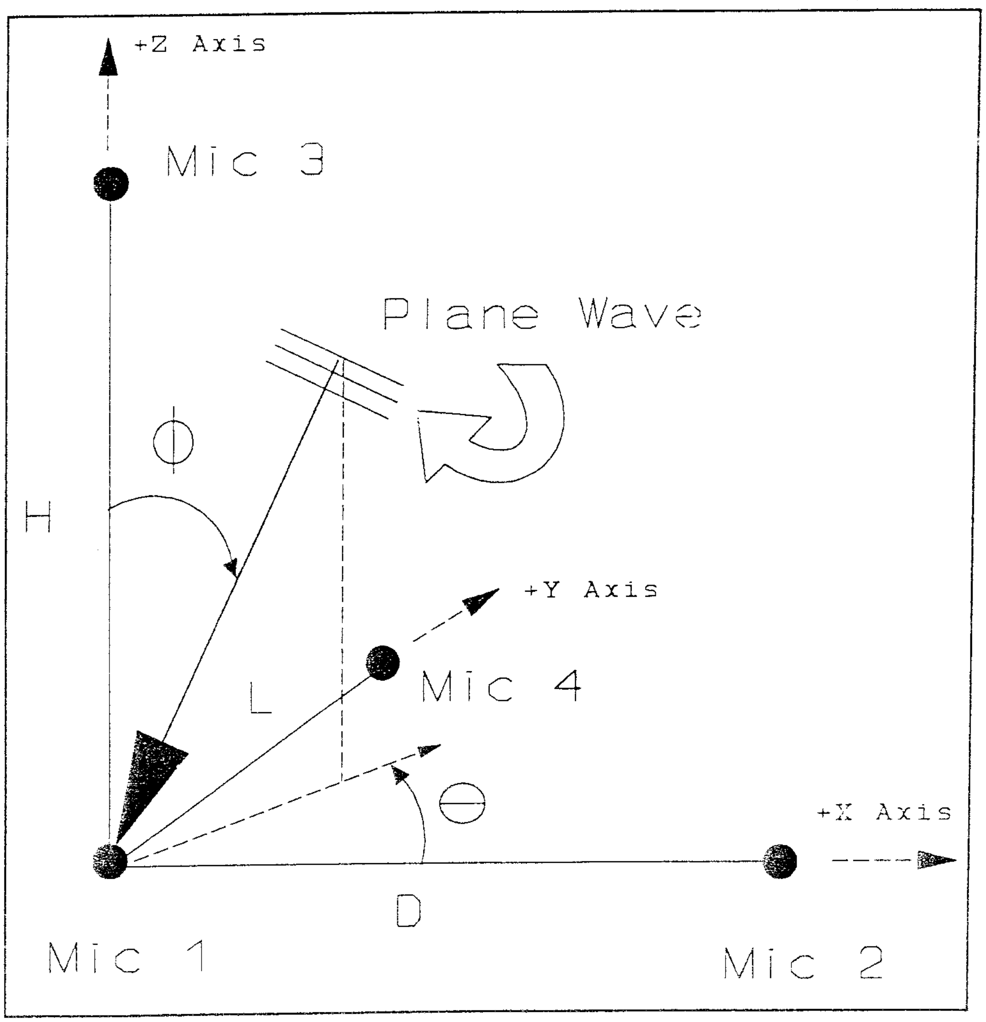
35. **Doppler Shifts.** It is well known that when a source and receiver move relative to each other at some constant subsonic speed, U, that the frequency of emitted sound, ω_0, shifts either up or down depending upon whether the source and receiver are moving away from each other or toward each other. If the speed of sound in the medium at rest is c_0 consider the following two questions:

a. Derive expressions for the perceived frequency for each of the two cases when the source and receiver are converging toward each other at U, and when they are diverging away from each other at U.

b. Derive expressions for the perceived frequency when the source and receiver are fixed in space relative to each other, but there is a mean flow of velocity U, which in one case is in the direction from source to receiver, and in the other it goes from receiver to source.

36. The figure below shows a cartesian array of four microphone sensors. Consider senor 1 to be at the origin, sensor 2 a distance D meters away along the positive X-axis, sensor 3 is H meters up the Z axis, and sensor 4 is L meters out the positive Y-axis. A plane wave from a distant source propagates past the array at some arbituary pair of angles θ, and ϕ. The angle θ is measured counter-clockwise from the positive X axis in the X-Y plane as seen in the figure. The angle ϕ is measured from the positive Z axis. Assume only one frequency for the plane wave where the wavelength is larger than twice the longest dimension of the array.

Solve for the speed of sound in terms of the array dimensions and the measured time or phase differences observed at the array sensors.



37. A harmonic plane wave of peak amplitude 0.1 Pa and frequency 300 Hz is incident in air at a 30° angle from normal onto a surface with specific acoustic impedance $Z = (2 + i\ 6)\rho_{air} c_{air}$. Assume the surface is in the plane $z = 0$, infinite in the x and y directions, and that c_{air}=343 m/s and ρ_{air}= 1.21 kg/m^3.

a.) What is the absorption coefficient?

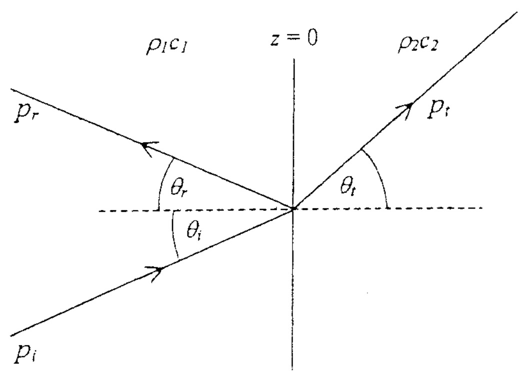
b.) Assuming that the problem is two-dimensional in the $x - z$ plane and the y coordinate can be neglected, write down an explicit expression for the pressure in the $x - z$ plane.

38. A spherical sound wave in free space loses amplitude as it propagates through air from the source. At low frequencies, the primary mechanism for this loss is "spherical spreading." That is, for every doubling of distance, the pressure amplitude is reduced by one half. At high frequencies, however, an additional three mechanisms contribute to the attenuation of the sound. These are related to the conversion of acoustic energy into heat. Name and describe briefly these three mechanisms.

39. Sound travels faster in warmer air. Suppose you have a point source and a distant point receiver with a very hot object in between them. The medium is reflection free. The hot object blocks the line of sight from source to receiver. Sketch the sound rays between source and receiver showing the expected refraction.

40. The diagram below shows the propagation vectors representing the incidence, reflection, and transmission of plane waves incident at an infinite plane interface separating media of different characteristic impedances. In this diagram, $\theta_t > \theta_i$. The condition(s) matching this situation is (are)

a) $c_2 > c_1$ b) $\rho_2 > \rho_1$ c) $\rho_2 c_2 > \rho_1 c_1$ d) $c_1 > c_2$ e) a and b



41. A very long circular duct of radius 50 cm with rigid walls is filled with air. A point source at the center of the duct (x=0, r=0) vibrates at a frequency of 500 Hz. Some modes are excited in the duct, and many are not. Give the cut-off frequencies in hertz for the modes that are propagating. For your convenience the values of selected Bessel functions are provided in the Table 1 output from Mathematica.

```
table1 = Table[{x, BesselJ[0, x], BesselJ[1, x],
    BesselJ[2, x], BesselJ[3, x], BesselJ[4, x]}, {x, 0, 8, 0.1}];

table1 // MatrixForm
```

0	1	0	0	0	0
0.1	0.997502	0.0499375	0.00124896	0.0000208203	2.60286×10^{-7}
0.2	0.990025	0.0995008	0.00498335	0.00016625	4.15834×10^{-6}
0.3	0.977626	0.148319	0.0111659	0.000559343	0.000020999
0.4	0.960398	0.196027	0.0197347	0.00132005	0.0000661351
0.5	0.93847	0.242268	0.030604	0.00256373	0.000160736
0.6	0.912005	0.286701	0.0436651	0.00439966	0.00033147
0.7	0.881201	0.328996	0.0587869	0.00692965	0.000610097
0.8	0.846287	0.368842	0.0758178	0.0102468	0.00103298
0.9	0.807524	0.40595	0.0945863	0.014434	0.00164055
1.	0.765198	0.440051	0.114903	0.0195634	0.00247664
1.1	0.719622	0.470902	0.136564	0.0256945	0.00358782
1.2	0.671133	0.498289	0.159349	0.0328743	0.00502267
1.3	0.620086	0.522023	0.183027	0.0411358	0.00683096
1.4	0.566855	0.541948	0.207356	0.0504977	0.00906287
1.5	0.511828	0.557937	0.232088	0.060964	0.0117681
1.6	0.455402	0.569896	0.256968	0.0725234	0.0149952
1.7	0.397985	0.577765	0.281739	0.0851499	0.0187902
1.8	0.339986	0.581517	0.306144	0.098802	0.0231965
1.9	0.281819	0.581157	0.329926	0.113423	0.0282535
2.	0.223891	0.576725	0.352834	0.128943	0.0339957
2.1	0.166607	0.568292	0.374624	0.145277	0.0404526
2.2	0.110362	0.555963	0.395059	0.162325	0.0476471
2.3	0.0555398	0.539873	0.413915	0.179979	0.0555957
2.4	0.00250768	0.520185	0.43098	0.198115	0.064307
2.5	-0.0483838	0.497094	0.446059	0.2166	0.0737819
2.6	-0.096805	0.470818	0.458973	0.235294	0.0840129
2.7	-0.142449	0.441601	0.469562	0.254045	0.0949836
2.8	-0.185036	0.409709	0.477685	0.272699	0.106669
2.9	-0.224312	0.375427	0.483227	0.291093	0.119033
3.	-0.260052	0.339059	0.486091	0.309063	0.132034
3.1	-0.292064	0.300921	0.486207	0.326443	0.145618
3.2	-0.320188	0.261343	0.483528	0.343066	0.159722
3.3	-0.344296	0.220663	0.478032	0.358769	0.174275
3.4	-0.364296	0.179226	0.469723	0.373389	0.189199
3.5	-0.380128	0.137378	0.458629	0.38677	0.204405
3.6	-0.391769	0.0954655	0.444805	0.398763	0.219799
3.7	-0.39923	0.053834	0.42833	0.409225	0.235279
3.8	-0.402556	0.012821	0.409304	0.418026	0.250736
3.9	-0.401826	-0.027244	0.387855	0.425044	0.266059
4.	-0.39715	-0.0660433	0.364128	0.430171	0.281129
4.1	-0.38867	-0.103273	0.338292	0.433315	0.295827
4.2	-0.376557	-0.138647	0.310535	0.434394	0.310029
4.3	-0.361011	-0.171897	0.281059	0.433347	0.323611
4.4	-0.342257	-0.202776	0.250086	0.430127	0.33645
4.5	-0.320543	-0.23106	0.217849	0.424704	0.348423
4.6	-0.296138	-0.256553	0.184593	0.417069	0.359409
4.7	-0.269331	-0.279081	0.150573	0.407228	0.369292
4.8	-0.240425	-0.2985	0.11605	0.395209	0.37796
4.9	-0.209738	-0.314695	0.0812915	0.381055	0.385307
5.	-0.177597	-0.327579	0.0465651	0.364831	0.391232
5.1	-0.144335	-0.337097	0.0121398	0.346619	0.395647
5.2	-0.11029	-0.343223	-0.0217184	0.326517	0.398468
5.3	-0.0758031	-0.345961	-0.0547481	0.304641	0.399625
5.4	-0.0412101	-0.345345	-0.0866954	0.281126	0.399058
5.5	-0.00684387	-0.341438	-0.117315	0.256118	0.396717
5.6	0.0269709	-0.334333	-0.146375	0.229779	0.392567
5.7	0.05992	-0.324148	-0.173656	0.202284	0.386586
5.8	0.0917026	-0.311028	-0.198954	0.173818	0.378766
5.9	0.122033	-0.295142	-0.222082	0.144579	0.369111
6.	0.150645	-0.276684	-0.242873	0.114768	0.357642
6.1	0.177291	-0.255865	-0.261182	0.0845982	0.344393
6.2	0.201747	-0.232917	-0.276882	0.0542833	0.329414
6.3	0.223812	-0.208087	-0.289871	0.0240416	0.312768
6.4	0.243311	-0.181638	-0.300072	-0.00590769	0.294534
6.5	0.260095	-0.153841	-0.30743	-0.0353466	0.274803
6.6	0.274043	-0.12498	-0.311916	-0.0640599	0.25368
6.7	0.285065	-0.0953421	-0.313525	-0.091837	0.231283
6.8	0.293096	-0.0652187	-0.312278	-0.118474	0.207742
6.9	0.298102	-0.0349021	-0.308219	-0.143775	0.183197
7.	0.300079	-0.00468282	-0.301417	-0.167556	0.157798
7.1	0.299051	0.0251533	-0.291966	-0.189641	0.131706
7.2	0.295071	0.0543274	-0.27998	-0.209872	0.105087
7.3	0.288217	0.0825704	-0.265595	-0.228102	0.0781139
7.4	0.278596	0.109625	-0.248968	-0.244202	0.050966
7.5	0.26634	0.135248	-0.230273	-0.258061	0.0238247
7.6	0.251602	0.159214	-0.209703	-0.269584	-0.00312601
7.7	0.234559	0.181313	-0.187465	-0.278697	-0.0297016
7.8	0.215408	0.201357	-0.163778	-0.285346	-0.0557187
7.9	0.194362	0.219179	-0.138873	-0.289495	-0.0809963
8.	0.171651	0.234636	-0.112992	-0.291132	-0.105357

42. The input specific acoustic impedance of a duct is measured at a frequency of 100 Hz, and found to be given by Z_{mo} = -j 6.0 Pa s/m (assuming $e^{j\omega t}$ time dependence). The duct has a length of 1.0 m. Assume the fluid in the duct is air, with $\rho_0 c$ = 415 Pa s/m, and c = 343 m/s.

a) Determine the termination specific acoustic impedance of the duct, Z_{mL}.
b) Using the results from part a), determine the pressure reflection coefficient. Since this is a complex quantity, put it in the form $\tilde{R} = |R|e^{j\theta}$.
c) What is the standing wave ratio for this duct?

43. Consider two semi-infinite fluid media in contact at a plane boundary located at $x = 0$. A plane wave in medium 1 is incident on the boundary at an angle of 45° to the normal, with a sound pressure level of 65 dB *re* 20 μPa. The fluid density of the incident medium is 0.09 kg/m^3 and the sound velocity is 1270 m/s. It is found that the transmitted pressure leaves at an angle of 11°, and the fluid density of medium 2 is 1.21 kg/m^3.

a) What is the sound pressure level of the transmitted pressure wave?

b) It is desired that no pressure should reflect from the interface. What should the incident angle be?

44. A plane wave in a fluid, of the form $p_i = P_o e^{jk_1\left(c_1 t - \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)}$, is incident upon a plane interface located at y = 0. The fluid on the incident side of the interface is characterized by density ρ_1 and phase velocity c_1, while the fluid on the transmitted side of the interface is characterized by density ρ_2 and phase velocity c_2.

a) Sketch the appropriate directions of the incident and reflected pressure on a two-dimensional plot. Justify your sketch.

b) If the transmitted pressure propagates at an angle of 30° to the interface normal, what is the value of c_2, relative to c_1?

c) If $\frac{\rho_2}{\rho_1} = 3\sqrt{3}$, what is the amplitude of the transmitted pressure, relative to the incident pressure?

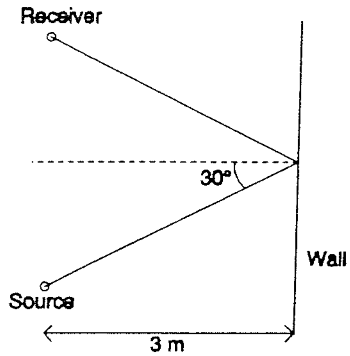
d) Assuming no viscosity, determine expressions for the tangential particle velocity components right next to the boundary in fluids 1 and 2.

e) If one wishes to achieve a condition of zero reflection from the interface, what should the incident angle be?

45. A monopole source is located 3 m from the only non-anechoic wall in a room. This wall has a transmission loss of 5 dB. The sound pressure level measured at the location 3 m from the wall, shown below, is 100 dB re 20 μPa.

a) What is the source power level in dB re 10^{-12} Watts?

b) What would the sound pressure level be if the transmission loss for the wall is 0 dB?



46. A plane wave of amplitude 0.17 Pa is normally incident on an impedance boundary with a specific acoustic impedance $Z = \rho_o c(1 + i)$ at a frequency of ω_2 rad/s. Here the ambient acoustic speed of sound is c and the ambient density is ρ_o. The magnitude of the wave reflected from the surface is B Pa. The $e^{-i\omega_2 t}$ time convention is assumed.

a) Determine an expression for B.

b) Clearly and quantitatively describe how a vibrating rigid wall can radiate a plane wave with a magnitude of B Pa at ω_2 rad/s into a fluid with ambient acoustic speed of sound c and ambient density ρ_o.

47. a) A source generates plane waves in water that propagate in the x-z plane with a pressure amplitude of A. The waves impinge on a water/air interface ($z = 0$) at an angle of 45°. Obtain an expression for the pressure transmitted into the air at any location (x,z). (Assume $c_{air} = 343$ m/s, $c_{water} = 1500$ m/s, $(\rho c)_{air} = 415$ Pa s/m, and $(\rho c)_{water} = 1.5\times10^6$ Pa s/m.)

b) The source now generates plane wave in air that impinge on the air/water interface at the same angle of 45°. Obtain an expression for the pressure transmitted into the water at any location (x,z). Explain any similarities or differences between the two results.

48. Suppose the velocity potential is given by

$$\Phi = 10^{-3} \cos\left(\frac{\pi}{3}x\right) \cos\left(\frac{2\pi}{3}y\right) e^{-i\omega_o t}$$

in MKS units in a rectangular room with 6 rigid walls ($0 \le x \le 3$, $0 \le y \le 3$, $0 \le z \le 4$ m). The field has no variation with respect to z. Here $\bar{v} = \nabla\Phi$ where $\bar{v}$ is the particle velocity vector.

a). Describe specifically where there are pressure nodes in the room.

b). What is ω_o? If you are not familiar with room acoustics you may wish to find $\bar{v}$, and solve the problem as a 2-D rectangular membrane.

c). What is the total energy in this room mode? Give the proper MKS units. Assume c_{air}=343 m/s and ρ_{air}= 1.21 kg/m^3.

A plane acoustic wave is incident on the ocean floor at an angle of 76°. The ocean floor is coarse silt having a density of 1790 kg/m^3 and a characteristic impedance of 2.76 x 10^6 Pa-s/m. The density and sound speed of seawater are 1026 kg/m^3 and 1500 m/s, respectively.

49. Calculate the angle of transmission.

50. If the incident wave has a peak pressure amplitude of 50 Pa, what is the SPL (re 1 μPa) of the reflected wave?

51. What is the SPL (re 1 μPa) of the transmitted wave?

52. At time $t = 0$ a transient spherically symmetric pulse is propagating *radially inwards* toward the origin. The acoustic pressure versus radial distance r at time $t = 0$ is given by

$$p = 0 \qquad \text{for} \quad 0 < r < R_0$$

$$p = A\frac{R_0}{r}\sin\left(\frac{\pi(r - R_0)}{R_0}\right) \quad \text{for} \quad R_0 < r < 2R_0$$

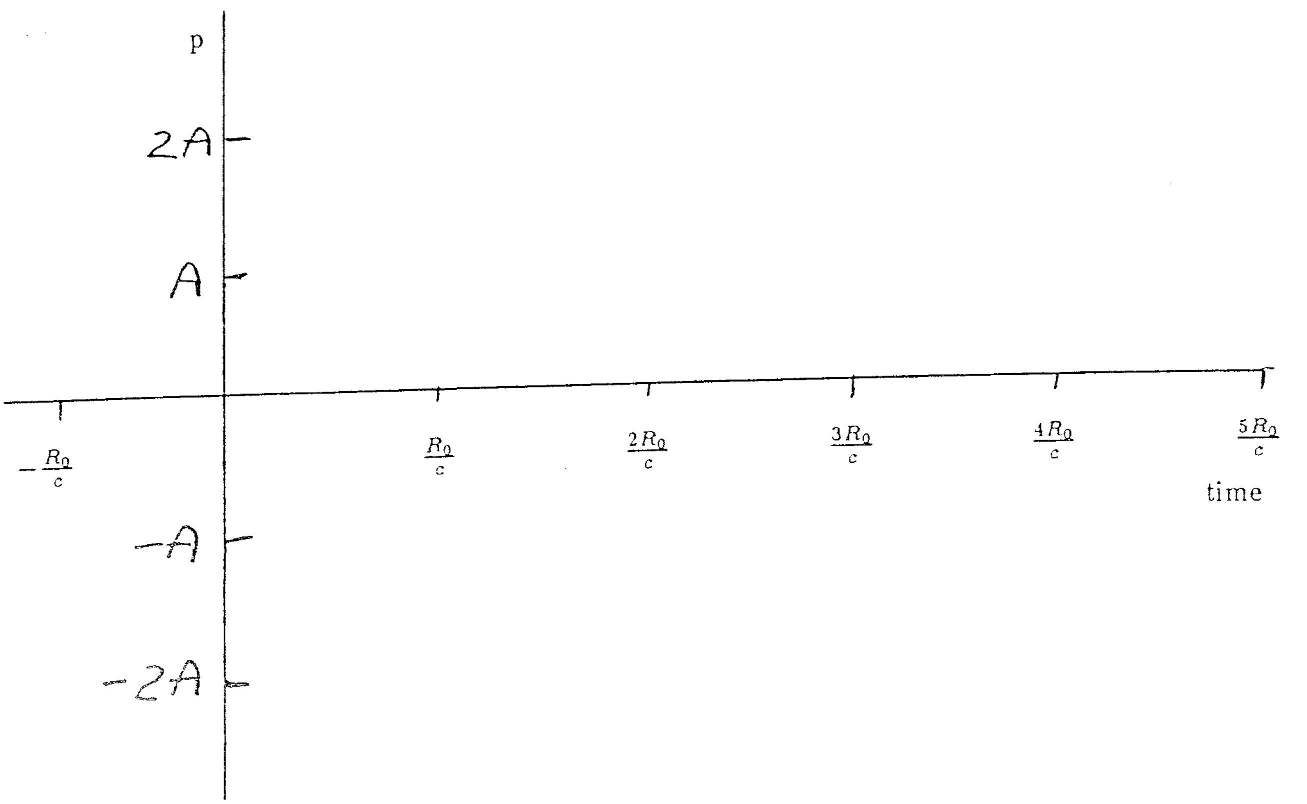
$$p = 0 \qquad \text{for} \quad r > 2R_0$$

Here A and R_o are positive constants. The medium is homogeneous, unbounded, and has ambient density ρ_o and sound speed c. Assume that the amplitude A is sufficiently small that nonlinear effects can be neglected for all points and for all times. There is nothing special about the coordinate origin, except that it happens to be the point toward which the incident spherical wave is initially converging.

a) Determine and sketch the acoustic pressure versus time at the origin for times between $-R_0/c$ and $5R_0/c$.

b) Determine and sketch the acoustic pressure versus time at $r = R_0$ for times between $-R_0/c$ and $5R_0/c$.

Give your sketches on graphs with scales like those shown below.



53. - Consider a spherically propagating wave which is converging towards the origin.

a) If at r = 100 m, the pressure due to the converging wave is found to be given by $p(100,t) = 0.01e^{j\omega t}$, write a general expression for the pressure field produced by this converging wave (i.e. neglect any diverging waves).

b) Determine an expression for the particle velocity associated with the converging wave.

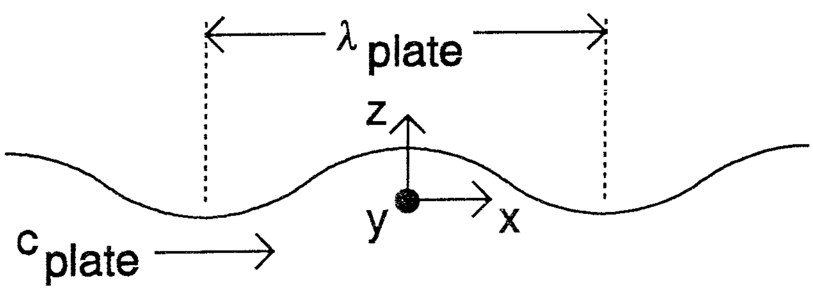
c) Obtain a general expression for the specific acoustic impedance associated with this wave.

d) What is the acoustic intensity associated with the wave?

e) For this converging wave (and assuming there is no source at the origin), what will be the form of the reflected wave which diverges from the origin? Justify your answer.

54. A sinusoidal bending wave with wavelength λ_{plate} = 0.1 m propagates along an infinite plate lying in the x - y plane at a speed of c_{plate} = a.) 200 m/s or b.) 500 m/s. The bending wave moves in the x direction and the problem is independent of the y direction, as shown in the diagram. Consider a light fluid, air, to be on the top side of the plate ($z > 0$), and a vacuum to be on the other side of the plate. Assume that ρ_{air} = 1.21 kg/m^3 and c_{air} = 343 m/s.

Give an expression for the acoustic pressure field as a function of position and time for $z > 0$ for both cases a.) and b.) above. Your answer should be correct up to an arbitrary amplitude and phase constant. (You do not have enough information to determine what this constant is.)



55. A device for determining if a water layer has formed at the bottom of a fuel tank is based on plane-wave acoustic reflection and transmission. Assume the fuel oil to have a sound velocity of 1350 m/s and density of 850 kg/m3. Water has a velocity of 1481 m/s and density of 998 kg/m3. At what angles relative to the normal to a plane signifying the oil-water interface would you place an acoustic transmitter and receiver to determine if a water layer is present and why?

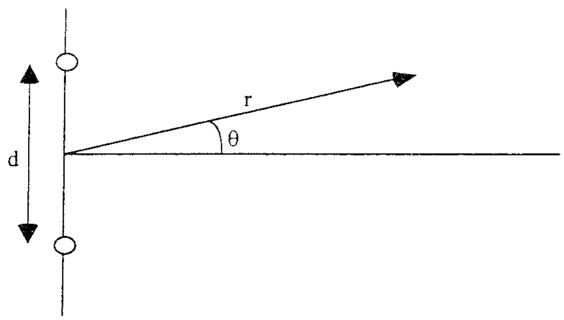
56. What are the boundary conditions used to analyze reflection and transmission of plane waves from plane interfaces? Provide both an equation and a very brief physical interpretation for each boundary condition.

57. A baffled circular piston of radius 0.1 m and of mass 1 kg is supported by a spring of stiffness 4×10^6 N/m. A plane wave with a freefield pressure level of 120 dB re 20μPa and frequency of 1 kHz is normally incident on the piston. Assume that the pressure on the face of the piston can be approximated by taking the piston to be rigid. What is the vibration level in dB re 1g of the piston, where g is the acceleration due to gravity (9.81 m/s^2)?

58. You wish to measure the sound pressure level in a 2 m long duct, but the only transducer available is an accelerometer. The duct has a piston (whose surface area matches that of the duct) vibrating with uniform motion at one end, and a rigid termination at the other end. If you place the accelerometer on the piston, it is found that the peak acceleration is 14 m/s^2 at a frequency of 500 Hz. What is the sound pressure level at a distance of 1.25 m from the vibrating piston? (Assume standard conditions in air.)

59. Consider a rectangular membrane of dimensions 0.5 by 0.75 m under a uniform tension of 10^4 N. The surface density of the membrane is 0.2 kg/m^2 and all of the edges of the membrane are fixed. a) For an acoustic wave in air that is normally incident on the membrane with a constant pressure as a function of frequency, find the frequency at which the response of the membrane will be maximum. b) Now examine the case when the acoustic wave is obliquely incident. There will be a certain angle of incidence at which the next highest mode [compared to part a)] will be excited. Find both the frequency of this mode and the corresponding angle of incidence for which the membrane will be excited for this mode.

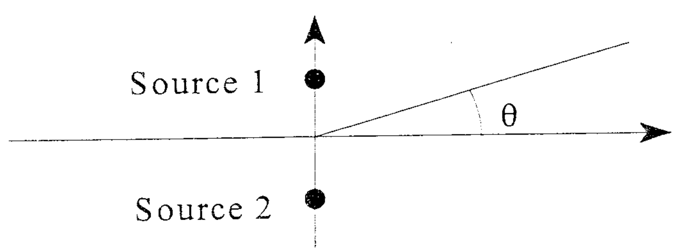
60. Two simple acoustic sources are separated by a distance d as shown in the figure below.



a) Write down an exact expression for the total acoustic pressure at any point.
b) Now, assume the amplitudes of the sources are equal and that they are driven with the same phase. Derive an expression for the acoustic pressure in the far field.
c) What is the far field directional factor?
d) Assume $kd = 3\pi$. At what angle(s) will the pressure amplitude be zero in the far field?
e) Sketch the beam pattern.

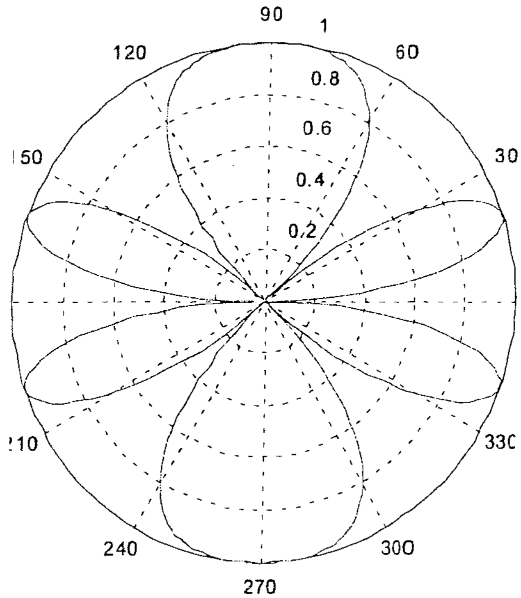
61. Find the far-field radiation pattern from a triangular-shaded, thin strip of length L mounted in an infinite, rigid baffle. Triangular shading means that the velocity has its peak value at the center of the strip and the falls off linearly with distance, reaching zero at the ends of the strip. Thin means the width of the strip is small compared to the wavelength.

62. Two simple acoustic sources are separated by a distance $d = 2\lambda$, where λ is the acoustic wavelength. The sources oscillate with equal amplitudes and frequencies. The phase of source 1 is shifted ahead of that of Source 2 by 60°.



a) Sketch the far-field radiation pattern generated by these sources.
b) Calculate the angles corresponding to the peaks of major lobes.

63. The far-field radiation pattern shown below is created by two simple sources separated by a distance d. The sources are aligned with the vertical axis of the figure and radiate with equal amplitudes and either a 0 or 180 degree phase difference. Determine the phase relation and the value of kd based on the radiation pattern.



64. A 10 cm diameter piston radiator is vibrating in a small closed box at 30 Hz. A small accelerometer is attached to the diaphragm and measures an rms acceleration of 0.1 g.

a) What is the rms sound pressure at 5m, 10m, 20m, and 40m?
b) What is the acoustic intensity at 10m?
c) At what frequency would you expect the direction from the piston source to matter in the measured radiated sound pressure?

65. Consider a simple source of sound radiating into free space:

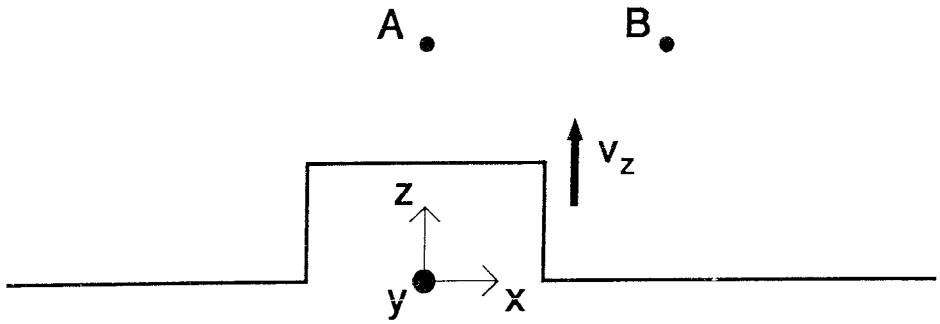
a) Compare the pressure radiated by this source when operated in standard air to that when operated with the same velocity amplitude and frequency in the Martian atmosphere. The characteristic acoustic impedance of the Martian atmosphere is a factor 100 smaller than that of standard air.

b) Compare the radiated acoustic powers.

66. An infinite plate with a single rectangular protrusion in it vibrates up and down in the z direction with a velocity of $v_z = V_o e^{-i2\pi f t}$. (The whole thing vibrates up and down, protrusion and all.) The protrusion is 0.5 cm wide and 0.25 cm high, and is centered around the origin. The plate and protrusion extend infinitely in the $\pm y$ directions. The fluid here is air with $\rho_{air} = 1.21$ kg/m^3 and $c_{air} = 343$ m/s. Assume the air is lossless, and the plate and protrusion have hard surfaces. The frequency f is fixed at 137.2 kHz. V_o is a constant.

a) What is the acoustic pressure field as a function of time at point A at (0, 0, 0.005) m? At point B at (0.005, 0, 0.005) m?

b) What is the specific radiation impedance (in the z direction) along the top of the protrusion?



67. A 1.0 cm radius spherical transducer, submerged in seawater, undergoes uniform radial oscillations at a frequency of 1 kHz. The peak displacement amplitude of the oscillation is 1.0 μm. The characteristic impedance of seawater is 1.54 x 10^6 Pa-s/m.

(a) What is the SPL (re 1 μPa) 10 m from the center of the transducer?

(b) How much power is radiated by the transducer?

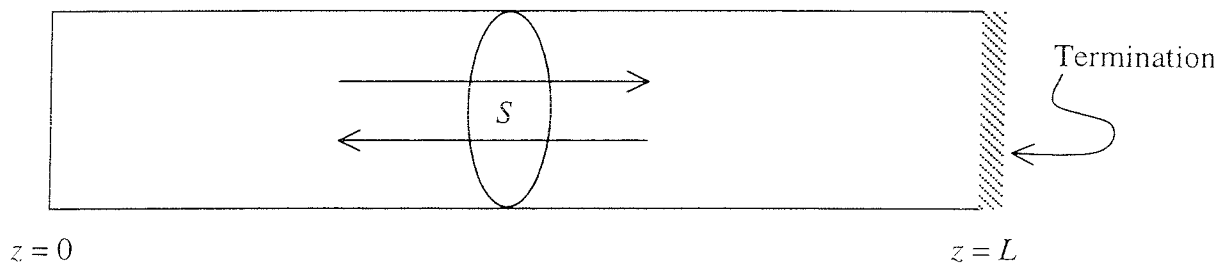
(c) What is the source strength?

68. A simple bubble in a liquid is in dynamic motion and radiates according to:

$$p(r) = \frac{\partial}{\partial t}\int_S \frac{\rho v_n}{4\pi r} dS \quad ,$$

where v_n is the normal surface velocity of the bubble, r is the far-field radius position, t is time, and S is the surface area of the bubble. Do an order-of-magnitude analysis of this equation for a single bubble of length scale a. This question is asking for an *approximate* expression for *the magnitude* of p at location r that depends upon the bubble radius and the surface velocity.

69. The figure below depicts two counter-propagating plane waves in a pipe of cross sectional area S and length L. The pipe is terminated with a load of unspecified mechanical impedance at $z = L$.



The total acoustic pressure in the pipe is given by the expression

$$p(z,t) = Ae^{j[\omega t + k(L-z)]} + Be^{j[\omega t - k(L-z)]},$$

where A and B are, in general, complex quantities.

a) If Z_{mL} is the mechanical impedance at $z = L$, show that

$$\frac{Z_{mL}}{\rho_o cS} = \frac{A+B}{A-B}.$$

b) Suppose that the pipe is terminated with a resistive load. What must the mechanical resistance be for there to be no reflection from the termination?

c) The input mechanical impedance (at $z = 0$) is given by

$$\frac{Z_{mo}}{\rho_o cS} = \frac{\dfrac{Z_{mL}}{\rho_o cS} + j\tan kL}{1 + j\dfrac{Z_{mL}}{\rho_o cS}\tan kL}.$$

What is Z_{mo} when the pipe is terminated with the resistive load found in part b)?

d) What are the resonance frequencies of the pipe terminated with the resistive load found in part b)?

70. A spherical source with uniform velocity, 0.3 m in diameter, is radiating sinusoidally in air; the resultant radiation is in free space. The acoustic power of this source is 10 kW at 1000 Hz. a.) Calculate the intensity, the sound pressure, and the acoustic particle velocity at 1 m and at 10 m from the source. b.) Calculate the phase angle (in degrees) between the sound pressure and particle velocity at distances of 0.5, 1.0, and 10 m from the source. What might you infer from these calculated phase angles?

71. A spherical source, 0.3 m in diameter, is radiating sinusoidally in air; the resultant radiation is in free space. The acoustic power of this source is 10 kWatts at 1000 Hz.

a) Calculate the intensity, the sound pressure, and the acoustic particle velocity at 1 m and at 10 m from the source.
b) Calculate the phase angle (in degrees) between the sound pressure and particle velocity at distances of 0.5, 1.0, and 10 m from the source. What might you infer from these calculated phase angles?

72. Consider the following expressions that describe the sound radiation from a simple point source (monopole) located at the origin $(0, 0, 0)$ of a Cartesian coordinate system, and from a dipole whose equal but opposite sources are located at $(0, \pm d/2, 0)$. The dipole expression was derived assuming far-field conditions and inter-source distance, $d << \lambda$, where λ is the acoustic wavelength.

For the monopole: $$p_m(r,t) = -\frac{ik\rho c Q}{4\pi r} e^{-i(\omega t - kr)}$$

For the dipole: $$p_d(r,\theta,t) = -\frac{k^2 \rho c Q d \sin\theta}{4\pi r} e^{-i(\omega t - kr)},$$

where Q is the source strength, θ is the angle measured between the x-axis and the vector from the origin to the field point (magnitude r), $k = \omega/c$, c is sound speed, ω is radian frequency, and t is time.

Show that the power radiated into free space by the dipole source, relative to the power radiated by the monopole source is:

$$\frac{\Pi_{dipole}}{\Pi_{source}} = \frac{8\pi^2}{3}\left(\frac{d}{\lambda}\right)^2.$$

73. A sphere oscillates radially in an anechoic room, with the radial displacement of the surface being of the form $\xi_r = \xi_0 e^{-i\omega t}$. The sphere has a radius of 1 m when $\xi_0 = 0$. The angular frequency ω is $2\pi f_0$, where f_0 is 500 Hz.

a) Suppose a receiver is at a position 4 m from the surface of the sphere. If $\xi_0 = 10^{-6}$ m, what is the sound intensity level at the receiver, using an appropriate reference level for air? Assume $\rho_{air} = 1.21$ kg/m^3, $c_{air} = 343$ m/s, and the air is lossless. Double check your math.

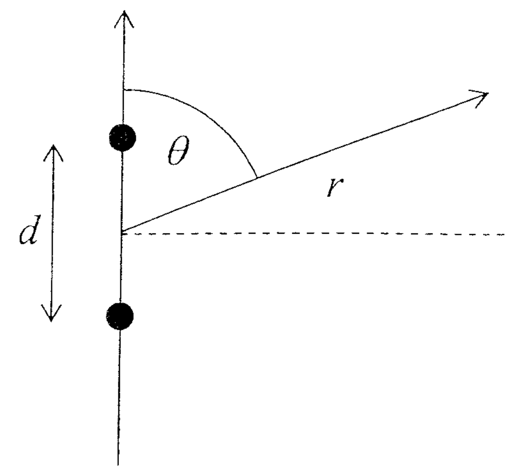
b) Assume that the sphere now oscillates radially at a very high frequency for a short finite duration. A second receiver is placed immediately adjacent to the surface of the receiver and senses a response beginning at $t = 0$ and ending at time $t = \Delta t$. No response is observed at this receiver before time $t = 0$ or after time $t = \Delta t$. What will be the duration of the received sound at the receiver described in part a)? Why?

74. A "thin," air-filled, and rigid sphere of diameter 0.1 m has a hole drilled into it of diameter d mm to make a Helmholtz resonator. What should the diameter of the hole be for a resonance of 320 Hz?

75. A circular piston situated in an infinite rigid, planar baffle oscillates harmonically in air. At a particular frequency, the angular half-width of the beam measured from the peak of the main lobe to the first null is 36°. If the frequency is doubled, what is the new half-width of the beam?

76. The far-field pressure radiated by a source composed of two simple acoustic sources of equal source strength and separated by distance d, is

$$P(r,\theta) = \frac{j\omega\rho_o Q}{2\pi} \cos\left(\frac{1}{2} kd \cos\theta\right) \frac{e^{-jkr}}{r}.$$



Show that the power radiated by this source is

$$W = 2W_S\left[1 + \text{sinc}(kd)\right],$$

where W_S is the power radiated by a single simple source of strength Q. [Note: This problem will be graded on the validity of the approach taken and your ability to work through the details. There are a number of steps involved in the solution. If you feel constrained by lack of time or get stuck on the details, be sure to explain your solution technique in as much detail as possible. This way, you will still get partial credit for understanding how to approach the problem].

77. A sphere of radius a is pulsating with a uniform surface velocity. What is its radiation efficiency, σ_{rad}, where

$$\sigma_{rad} = \frac{P}{\rho_0 c S \langle v^2 \rangle},$$

and P is the radiated power, S is the surface area, and $\langle v^2 \rangle$ is the time-space average of the squared velocity.

78. What is the acoustic intensity inside a tube of length L filled with air and driven at one end with a piston at 60 Hz? The amplitude of the displacement of the piston in 10^{-5} meters. The other end of the tube has an impedance of 250(1 + i) Pa-s/m.

79. Derive the equation for the specific acoustic impedance of the waves emanating from a point source in free space.

a) Discuss the reasons why this expression is complex.
b) What conditions must be met in order for this impedance to be predominantly real?
c) From the practical point of view, use the derived result to estimate a distance, in wavelengths, from the source where near-field effects are negligible (infinity is **not** the answer I'm looking for!).
d) What happens to the particle velocity as one measures it closer id closer to the source, assuming the sound pressure remains constant?
e) If you interpret this result such that r (radial distance) is the radius (a) of a spherical acoustic source, what is the likelihood of generating loud noises with a small source? Why?
f) For the simple point source, derive the equation for the acoustic intensity.
g) How does the active intensity (the real part of intensity) change with distance from the source? Explain the significance of this result.

80. A loudspeaker is place 1-m above the horizontal in an anechoic space. 3-m away, a perfectly reflecting rigid boundary is placed on the horizontal. Microphone "A" is 1-m above the edge of this floor. 4-m further into the reflecting floor boundary, microphone "B" is placed 1-m above the floor. Assume the reflecting floor is an infinite half-plane and the far boundaries in all other directions are anechoic.

The loudspeaker frequency response is measured using both microphones A and B. Then the loudspeaker and microphone B are switched in position and the loudspeaker response is measured again at microphones A and B. Explain any differences in the two measurements for each microphone.

81. **Propagation.** A 1 cm diameter circular ultrasonic transducer in a rigid baffle vibrates like a piston at 500 kHz and radiates into an enormous body of water ($\rho = 1000$ kg/m^3, $c = 1500$ m/s).

(a.) Along the axis of the transducer (z axis), what is the farthest distance (in m) from the transducer face at which a pressure null (p=0) is found? [You may have to do a little algebra to get this.]

(b.) How many λ's (acoustic wavelengths) is this pressure null from the transducer face?

82. An infinite plate located in the x-y plane is vibrating in air, such that the transverse velocity of the plate is described by

$$v_p(x,t) = V_o e^{j(\omega t - 5x)}$$

a) If the frequency of excitation is 500 Hz, find an expression for the pressure field in the air. (Assume the plate is located at z = 0, and we are interested in the half-space described by z > 0.)

b) If the frequency of excitation is changed to 100 Hz (but the expression given above for the plate velocity is still valid), determine the pressure field in the air.

c) Explain the results obtained in parts a) and b).

33. You have nine microphones that you can put in a line array. You want the width of the main lobe (defined as the total angular variation between the nulls on either side of the main lobe) to be no more than 15 degrees at frequencies above 3 kHz. a) Determine the space between the microphones. b) At what angle will aliasing occur, if at all? c) While maintaining the same aperture (total length of the array), what is the minimum number of microphones you can have without introducing aliasing?

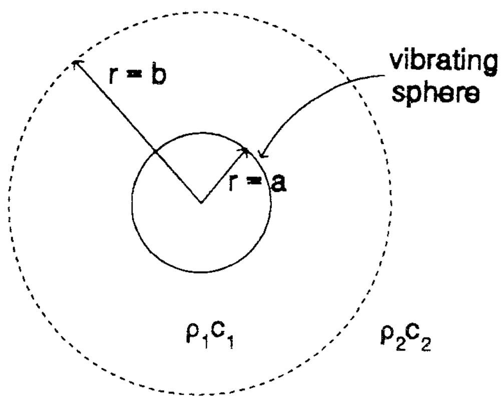
34. A very large plate, which is so large in the x and y directions that you may consider it of infinite extent, rests on the bottom of the ocean and is uniformly covered with a 0.75 m layer of quartz sand ooze. Assume that the plate vibrates harmonically with normal component $v_z = 0.001e^{-i\omega t}$ m/s at 500 Hz. The quartz sand ooze has a speed of sound of 1750 m/s and has an ambient density of 2000 kg/m^3. You may assume the water has a speed of sound of 1500 m/s and an ambient density of 1000 kg/m^3.

a) What will be the SPL of the acoustic pressure at the surface of the large plate? Since this is an underwater problem, use the reference value for SPL appropriate for water. (This is even though the plate is in contact with the ooze, for which we have no reference value.)

b) What will be the time average sound power (per unit area) radiated from the plate?

35. A train traveling at 26.82 m/s enters a long tunnel with the same cross sectional area and frontal shape as the train. Determine the magnitude of the pressure wave created in the tunnel. Assume 1-D pressure propagation in the tunnel, assume frictional effects between the train and the tunnel wall are non-existent, and assume the speed of sound in air is 340 m/s with air density 1.2 kg/m^3.

36. A sphere of radius a has a uniform velocity v_s at a frequency ω. It is surrounded by a fluid of density ρ_1, and sound speed c_1. At r = b, where b > a, is an interface between fluid 1 (ρ_1 c_1) and fluid 2 (ρ_2 c_2). Derive an expression for the power of the acoustic radiation.



37. A train traveling at 60 mph enters a long tunnel with the same cross sectional area and frontal shape as the train (assume frictional effects between the train and tunnel wall are non-existent). Determine the magnitude of the pressure wave created in the tunnel.

88. The circular hole in a 1-inch thick wall of a cinder block has a radius of 3/4-inches. The volume of the cavity inside the block is 4x4x6-inches. a) What is the Helmholtz resonance frequency of this hole/cavity combination? b) What must the radius be changed to in order to have a resonance frequency of 500 Hz?

89. Loads imposed by winds on missiles resting on their launching pads sometimes cause serious stresses in the structure. If we model this situation as a 2-D cylinder of length, L, mass, m, and diameter, D that is supported simply at both ends, then it is possible to compute the bending displacement response for specific idealized forcing functions. Do so for the case of coherent vortex shedding from the entire length of the cylinder. Here, the time-dependent transverse force is given by:

$$f(t) = \frac{\rho U^2}{2} D C_1 \cos\Omega t \quad ,$$

where C_1 is a constant, ρ is the air density, Ω is the radian frequency of vortex shedding = 2π (0.2 U/D) rad/s, and U is the constant wind velocity. Assume a constant bending rigidity, EI, and ignore structural damping. Lay out the solution to this problem starting with the differential equation:

$$EI\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} = f(t)$$

where w is the displacement. Specify the boundary conditions, and the methods of solution, with a description of: 1) the homogeneous solution, and 2) the particular solution. Go as far as you can to a final answer spending no more than one (1) hour on this problem.

90. The measured loss factor (a measure of the total energy dissipated) of a 1/16-in steel plate (ρ = 7700 kg/m^3) is η = 0.01 at 1kHz. One side of the plate is in water (ρc = 1.5×10^6) with no fluid (e.g. air, the effect of which you may ignore) on the other side. Assume that the internal energy losses in the plate can be ignored. Estimate the radiation efficiency, σ_{rad}. Equations that may help are

$$\sigma_{rad} = \frac{P_{rad}}{\rho_o c S\langle v^2\rangle}$$

where P_{rad} is the radiated power, and

$$\eta = \frac{P_{loss}}{\omega E}$$

where P_{loss} is the power loss.

91. **Air column over water.** A vibrating tuning fork of frequency 384 Hz is held over the end of a vertical glass tube. The other end of the tube dips into water. The temperature of the air and the water is 23 °C. Resonance occurs when the top of the tube is 21.9 cm and also 66.4 cm above the water surface.

a. Calculate the speed of sound of the air in the tube.
b. Calculate the end correction (*i.e.*, additional effective length) for the tube when the air is vibrating in the "21.9 cm mode" of the tube.
c. What would be the speed of sound if the temperature of the air and water were reduced to 7 °C?

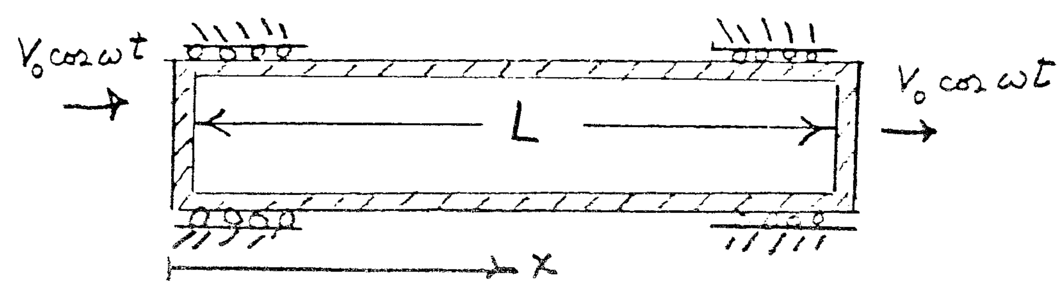
92. An open organ pipe has a radius of 1 ft. a) Determine the length required for a 100-Hz tone. b) What is the power radiated by the pipe if the velocity of the open end is 0.1 m/s at 100 Hz? The effective length of an unflanged pipe is L + 0.6a and the radiation impedance is $\rho_o cS$ [1/4 $(ka)^2$ + i 0.6 ka].

93. A hollow tube (filled with fluid) with rigid walls and enclosed at both ends has a total interior length L. The fluid has sound speed c and ambient density ρ. The tube is aligned with its axis along the x-axis and is oscillated back and forth with angular frequency ω, so that the velocity of either end in the $+x$-direction is given by $V_0 \cos \omega t$, where $V_0/\omega << L$, and $V_0 << c$. Assume that all transients have died out, so the disturbance (if there is one) in the tube is of a steady-state form (acoustic pressure p oscillating also with angular frequency ω; also assume that viscosity and gravity have negligible influence on the steady-state disturbance. The x-origin coincides with the cavity's left end in its average position.

(a) Determine an expression $\hat{p}(x)$ for the (complex) amplitude of the acoustic pressure in the tube.

(b) Give an expression for the acoustic pressure $p(L, t)$ at the end $x = L$.

(c) At what values of ω would one encounter resonance?

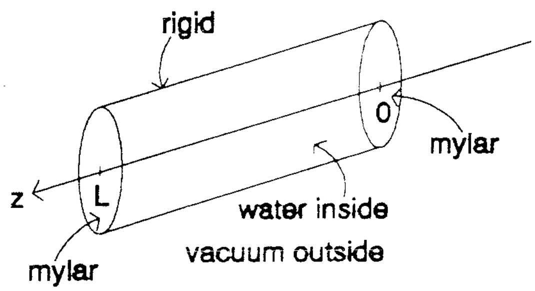


Q.4. Consider a rigid circular tube of length L floating in the vacuum of outer space. The two ends of the tube are covered with thin sheets of Mylar, an acoustically transparent material. The tube is completely filled with water, which we assume will not freeze. We will further assume that the cross-sectional dimensions of the tube are such that the acoustic field can be treated as a one-dimensional field. (i.e. The acoustic variables are only a function of z.) Using the appropriate speed of sound c and density ρ_0 for water, and using appropriate boundary conditions at the two ends of the tube at $z = 0$ and $z = L$,

a) Calculate the pressure modes of the tube in the z direction.

b) Calculate the characteristic frequencies for these modes.

c) Show your work and make an attempt to justify your assumptions.



95. A *very* small (radius much smaller than a wavelength) spherically symmetric source produces sound of constant angular frequency ω in an unbounded fluid (sound speed c and ambient density ρ_0). The source radiates a time averaged acoustic power $\mathcal{P}_{\rm av}$. It is observed that the radial component of the fluid velocity very close to the source ($r << c/\omega$) is such that

$$r^2 v_r \approx K \cos(\omega t - \phi_0)$$

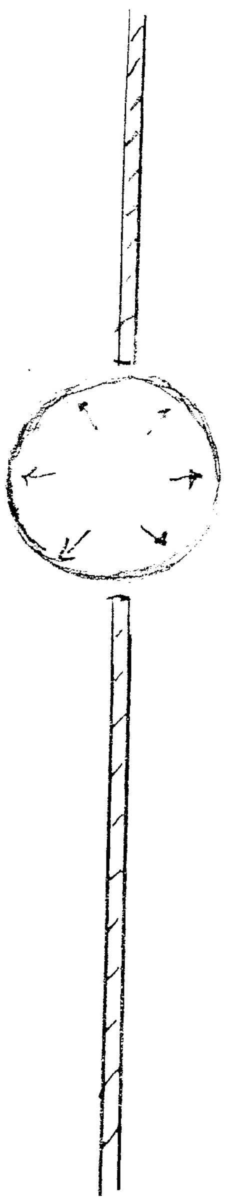
where K and ϕ_0 are constants. Determine K in terms of ω, $\mathcal{P}_{\rm av}$, c, and ρ_0.

96. PULSATING SPHERE: A 0.1 m radius sphere is submerged in sea water and pulsates with a uniform radial velocity at a frequency of 10,000 Hz, producing a peak pressure of 100 Pa at a range of 100 m.

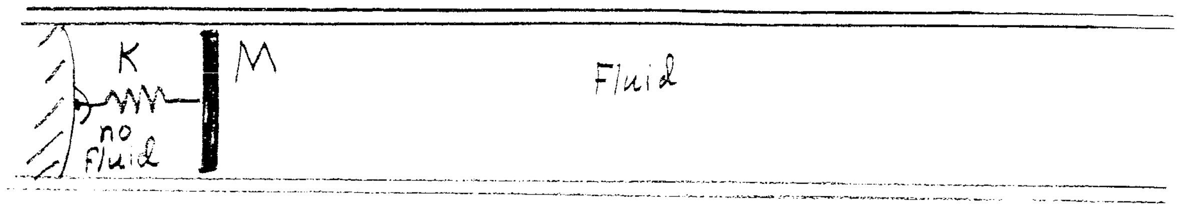
(a) Determine the acoustic power (in watts) radiated by the sphere.

(b) Determine the amplitude of the surface velocity on the sphere.

(c) Determine the acoustic intensity (watts per square meter) at a horizontal distance of 100 m if the sphere (vibrating with the uniform radial velocity determined above) is suspended within a hole of radius slightly larger than 0.1 m in a vertical thin rigid sheet, as illustrated below.



97. A flat 0.2 m diameter loudspeaker, with a diaphragm mass M of 15 grams and stiffness K of 3000 N/m is mounted at the end of an infinitely long rigid walled tube, as sketched in the figure below. Assume that the tube is filled with fluid with density ρ of 1.41 kg/m^3 and sound speed c of 320 m/s, and that the diaphragm of the loudspeaker exactly matches the cross-sectional area of the tube.



(a) An incident sound wave from the far end of the tube is of a frequency exactly equal to that of the loudspeaker in vacuum and has a sound pressure level (SPL) of 65 dB re 20μPa. What is the displacement amplitude of the diaphragm?

(b) Determine the time-averaged intensity of the total acoustic field (incident plus reflected) in the tube at a distance of 5 wavelengths from the nominal location of the diaphragm.

(c) Determine the distance from the face of the diaphragm where the acoustic pressure amplitude has its first minimum value. What is the value in pascals of this minimum value?

(d) Suppose that there is initially no acoustic disturbance in the tube and that the diaphragm is displaced outward from its equilibrium position an initial distance of 10 μm and then released. Determine the decay constant (nepers per second) and time interval between zero crossings of its subsequent vibration. Give a rough sketch of the diaphragm displacement as a function of time.

(e) What is the total acoustic energy radiated during the oscillations described in item (d) above?

98. The farfield pressure from a point dipole source is:

$$p = -ikD_z \cos\theta \frac{e^{ikR}}{R}$$

where D_z is the dipole-moment amplitude of the source, θ is the direction from the source, k is the acoustic wavenumber, and R is the distance from the source to the farfield receiver.

a) What is the power radiated by the source?

b) If the source is placed at a distance z_s from a rigid wall, what is the power radiated?

c) If the source is placed at a distance z_s from a pressure release surface backed by a vacuum, what is the power radiated?

99. Consider one simple source in an infinite isotropic fluid medium. The acoustic pressure p radiated by this source is described as:

$$p(r) = A \frac{e^{-jkr}}{r}$$

where A is a convenient strength factor (incorporating the amplitude of the velocity of the source, its active surface area, frequency, and certain parameters of the medium), r is the radial distance from the center of the source to the field point and k is the acoustic wavenumber (= $2\pi f/c$ where f is frequency and c is the sound speed in the medium).

Case I: Consider two such identical, equistrength, in-phase, simple sources separated by center-to-center distance d. (a) Obtain an expression for the far field acoustic pressure as a function of the radial coordinate r now measured from the mid point of the line segment joining the two sources and the angle θ measured from that line segment. (b) What is the direction of maximum output for this source combination? is there more than one? (c) are there directions of zero output? if so discuss what that means? (d) if the parameter $kd \ll 1$, what is the radiated acoustic intensity compared to that of one source alone? (e) if the parameter $kd \ll 1$, how does the power output of each of the pair compare to its power output if it were by itself?

Case II: Consider that the two sources are now 180° out-of-phase with one another.

(f) Obtain an expression for the far field acoustic pressure as a function of r and θ as defined above. (g) what is (are) the direction(s) of maximum output? (h) are there directions of zero output? (i) if the parameter $kd \ll 1$, qualitatively compare the acoustic output of the pair versus that of one source alone.

(j) One source near an infinite plane rigid wall compares to which case above? Why?

(k) One source, underwater, but near the surface of the water compares to which case above? Why?

(l) Consider a small closed right circular cylinder (a tin can) which is vibrating. The cylindrical face has no radial motion but the circular ends are moving such that at any instant of time they vibrate in the same direction, either both up or both down. Which case above does this compare to?

100. Consider a piston within a tube of length L, such that the piston covers the entire cross-section of the tube. The end of the tube, at $x = L$, is open, which we will approximate as a pressure release boundary. The piston, at $x = 0$, vibrates such that the velocity of the piston, v_p, is given by:

$$v_p(t) = U_o \sin(\omega t) \quad ; \; 0 \le t \le \frac{2\pi}{\omega}$$
$$= 0 \quad ; \; \text{otherwise}$$

with $\omega = \dfrac{4\pi c}{L}$. Determine (and sketch) the pressure which would be measured in the tube as a function of time at the location $x = 7L/8$, for times between $t = 0$ and $t = 3L/c$.

101 Acoustic waves can propagate through a porous medium similar to the way they do through homogenous fluids. Such a porous medium may be characterized by a porosity Ω [dimensionless] which gives the fraction of volume not occupied by the solid that the fluid can occupy, a flow resistance Φ [kg / (s m^3)] to the propagation through the medium, and an empirical constant K [dimensionless] to describe the twisty path that the fluid must take through the pores of the medium.

For such a porous medium it can be shown that the linearized one-dimensional equations of continuity, force balance, and state are:

$$\Omega \frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial u}{\partial x} = 0$$

$$\rho_o \frac{K}{\Omega} \frac{\partial u}{\partial t} + \Phi u + \frac{\partial p'}{\partial x} = 0$$

$$p' = \frac{c_o^2}{\gamma} \rho'$$

Here ρ', p', and u are the acoustic density, acoustic pressure, and average particle velocity in the x direction for the fluid occupying the pores of the medium. The variables ρ_o, c_o, and γ correspond to the ambient properties of the fluid.

a) Derive a wave equation for the average particle velocity, u.

b) A "plane wave" propagates in the medium as $u = \hat{u} e^{-i\omega t} = A e^{i(k_x x - \omega t)}$ where k_x is complex. What is k_x^2 ?

c) If $\Phi \to 0$, at what speed does u propagate through the porous medium?

102. If a spatially distributed and time varying source of heat is present in a nominally homogeneous and quiescent fluid, the linear acoustic equations are modified (with neglect of thermal conduction) to

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{\mathbf{v}}' = 0$$

$$\rho_0 \frac{\partial \vec{\mathbf{v}}'}{\partial t} = -\nabla p'$$

$$\frac{\partial p'}{\partial t} - c^2 \frac{\partial \rho'}{\partial t} = Bq$$

where B is a constant (as are c and ρ_0) and where $q(\vec{\mathbf{x}}, t)$ represents the heat added per unit volume and time. From these one can derive an *inhomogeneous wave equation* for acoustic pressure p' of the form

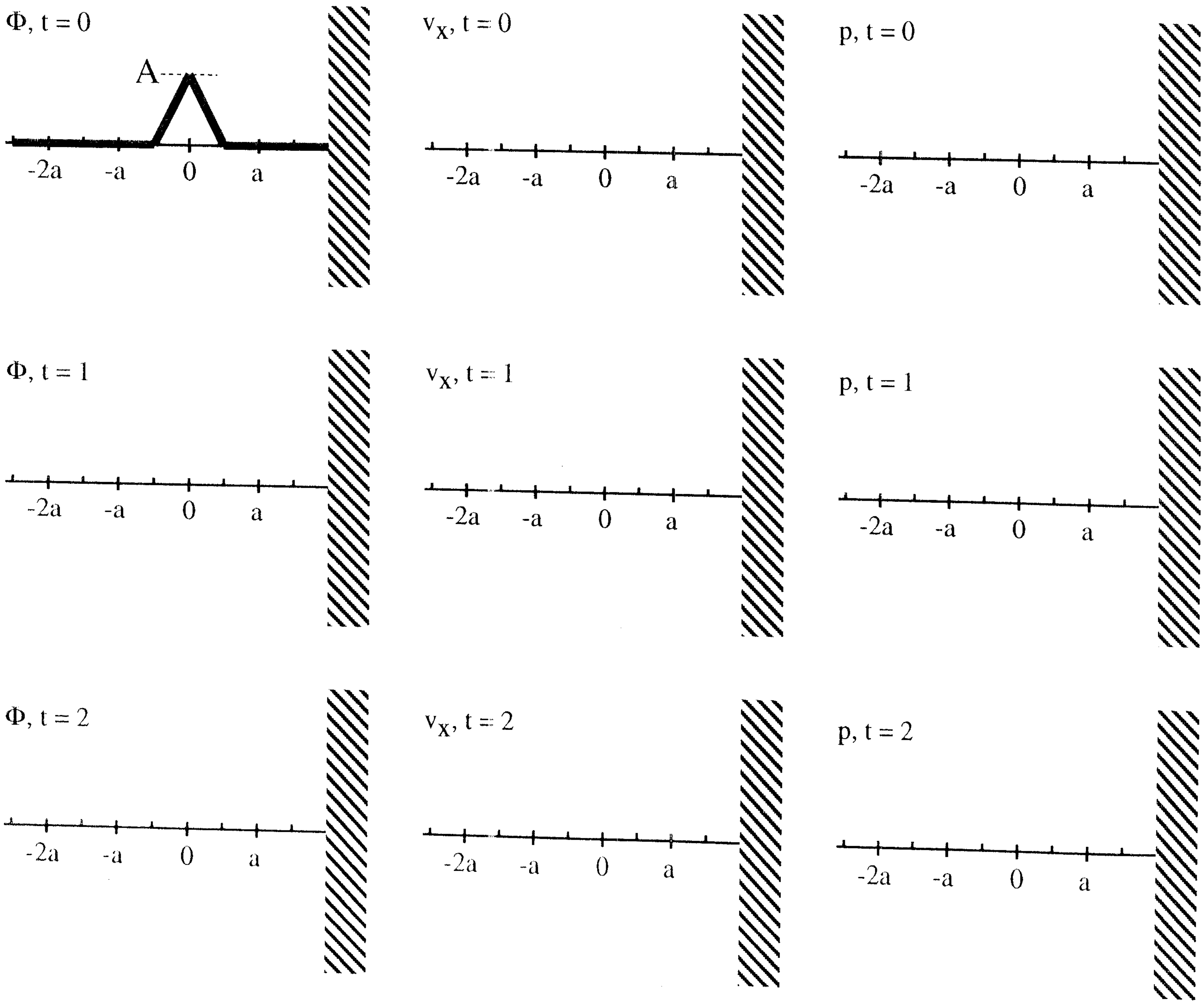
$$[\text{Linear operator acting on}]p' = [\text{another operator acting on}]q$$

What is the explicit mathematical form of this wave equation? (If you wish you can delete the primes on p' and $\vec{\mathbf{v}}'$ in your derivation.)

103. The velocity potential $\Phi(x)$ at time $t = 0$ is shown at the top left of Figure I. (next page) for a homogeneous, quiescent, isotropic fluid near a hard wall (the vertical, cross-hatched block). Assume that $\partial\Phi/\partial t = 0$ for all x at time $t = 0$. Let the speed of sound $= a$.

a) What is the correct boundary condition for Φ at a rigid wall?
b) In the appropriate places on Figure I., sketch $\Phi(x)$ at times $t = 1$ and $t = 2$ seconds, as well as the pressure $p(x)$ and the particle velocity in the x direction $v_x(x)$ at times $t = 0$, 1, and 2 seconds.

Figure I.


Φ, t = 0
A
-2a
-a
0
a
v_x, t = 0
-2a
-a
0
a
p, t = 0
-2a
-a
0
a
Φ, t = 1
-2a
-a
0
a
v_x, t = 1
-2a
-a
0
a
p, t = 1
-2a
-a
0
a
Φ, t = 2
-2a
-a
0
a
v_x, t = 2
-2a
-a
0
a
p, t = 2
-2a
-a
0
a

104 A set of linear acoustic equations obtained by Stokes (1845), which include the effects of viscosity and apply to sound waves at points substantially removed from solid surfaces, can be taken as

$$\frac{\partial p}{\partial t} + \rho c^2 \nabla \cdot \vec{v} = 0$$

$$\nabla \times \vec{v} = 0$$

$$\rho \frac{\partial \vec{v}}{\partial t} = -\nabla p + \frac{4}{3}\mu \nabla^2 \vec{v}$$

where ρ is the ambient density and c is the speed of sound. Here, μ is the viscosity and is a constant.

a) Derive a single partial differential equation for $p(\vec{x}, t)$ that does not include $\vec{v}(\vec{x}, t)$.

Hints: Assuming b is a scalar and $\vec{A}$ is a vector, the following relationships hold:

$$\nabla \times \nabla b = 0$$

$$\nabla \cdot \nabla \times \vec{A} = 0$$

$$\nabla^2 \vec{A} = \nabla \nabla \cdot \vec{A} - \nabla \times \nabla \times \vec{A}$$

b) What would be the wavenumber in the corresponding "Helmholtz equation"? It may be complex.

c) Suppose we define the velocity potential such that $\vec{v} = \nabla \Phi$. What will be the corresponding expression for p in terms of Φ? It is OK to make reasonable assumptions.

MATHEMATICS

1. A sinusoidal voltage,

$$V = E\sin(\omega t)$$

is passed through a half-wave rectifier which clips the negative portion of the wave. Find the Fourier series of the resulting periodic function.

2. One seeks to determine the work done by a force over various paths S, where the force and work are defined by

$$\vec{\mathbf{F}} = x\vec{\mathbf{i}} + 3y\vec{\mathbf{j}}$$

$$W = \int_S \vec{\mathbf{F}} \cdot d\vec{s}$$

and the paths to be considered are as follows

a) Evaluate the work done by the force along Path S_1: (0,0) to (2,0) to (2,1) along straight line segments.

b) Evaluate the work done by the force along Path S_2: (0,0) to (2,1) along a straight line.

c) Explain any discrepancy or similarity between the results of (a) and (b) in terms of general mathematical theorems.

d) Evaluate the work along Path S_3: (0,0) to (2,0) to (2,1) to (0,0) along straight line segments, in as brief a manner as you possibly can. (A non-brief derivation will count for nothing for this particular question.)

3. Evaluate the following, where i is the unit imaginary number:

a) $\ln(3+4i)$

b) $\sqrt[3]{1}$

c) $\left|\dfrac{a-ib}{b-ia}\right|$

4. Find the complex roots of the following equations:

(a) $\left(1-i\sqrt{3}\right)^{\frac{1}{2}}$

(b) $e^{2z} = 2i$

5. Find an expression for the real part of:

$$\frac{e^{-ix}}{1+e^{a+ib}}$$

6. Evaluate: $\sqrt[3]{-i}$

7. Consider the differential equation

$$y'' + k^2 y = 0 \quad k^2 > 0 \ .$$

Which of the following are *not* the general solution of this differential equation?

a. $y(x) = C_1 \cos kx + C_2 \sin kx$
b. $y(x) = C_1 e^{ikx} + C_2 e^{-ikx}$
c. $y(x) = C_1 \cos kx + C_2 e^{-ikx}$
d. $y(x) = C_1 \cosh kx + C_2 \sinh kx$
e. $y(x) = C_1 \cosh(ikx) + C_2 \sinh(ikx)$
f. $y(x) = C_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + C_2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

8. Let ξ be the particle displacement of a fluid. In one dimension, plane waves in this fluid obey

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}$$

where x is the spatial coordinate, t is time, and c is wave velocity. Write out **the most general form** of the solution to this equation. Then, demonstrate that this general solution does indeed describe waves propagating along the x-axis.

9. A dynamical system is goverened by the following differential equation and boundary conditions on the variable y (x):

$$y'' + k^2 y = F_o \delta(x - L/4) \qquad o \le x \le L$$

$$y(o) = o \qquad y'(L) = o$$

(a) Find the eigenvalues and eigenfunctions.

(b) Obtain the solution of the non-homogeneous system explicity.

10. Given the following recurrence formula for Legendre polynomials:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

and the integral result:

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \frac{2}{2n+1}\delta_{nm}$$

show that

$$\int_{-1}^{1} x^2 P_{n+1}(x)\,P_{n-1}(x)\,dx = \frac{2n(n+1)}{(4n^2-1)(2n+3)} \qquad \text{for all n}$$

11. In the following, $J_p(x)$ is the Bessel function of the first kind of order p. Given the recurrence formulae:

$$J_{p-1} + J_{p+1} = \frac{2p}{x}J_p \qquad \text{and} \qquad J_{p-1} - J_{p+1} = 2J_p'$$

and the integral relationship

$$J_0(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\theta)\,d\theta$$

a) Derive the result

$$J_1(x) = \frac{2}{\pi}\int_0^{\pi/2} \sin(x\sin\theta)\sin\theta\,d\theta$$

b) Integrate the right hand side of this last result by parts to obtain

$$x^{-1}J_1(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\theta)\cos^2\theta\,d\theta$$

12. Give a brief definition of each of the following mathematical terms or quantities:

a. homogeneous
b. linearly independent
c. analytic
d. radius of convergence
e. singular point
f. boundary value problem
g. power series

13. What are eigenvalues and eigenfunctions? Where do they come from? What are some of their properties? What can one do with them?

14. What does it mean to say a differential equation is linear?
15. What does it mean to say a differential equation is homogeneous?
16. What is the particular solution of a differential equation?
17. Is the particular solution unique?
18. Discuss some of the properties of a power series.
19. What does the term "radius of convergence" mean?

Solve $y'' + 4y = f(x)$ for the following cases:

20. $f(x) = 0$ with boundary conditions $y(0) = 0$ and $y(\pi/2) = 0$
21. $f(x) = x$ with boundary conditions $y(0) = 1$ and $y(\pi/2) = 2$
22. $f(x) = 0$ with boundary conditions $y(0) = 0$ and $y(\pi/4) = 0$
23. $f(x) = x$ with boundary conditions $y(0) = 1$ and $y(\pi/4) = 2$

24 **Power Series.** The D.E. governing the displacement of the harmonic oscillator as a function of time is

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

The solution, often taken for granted, is

$$x(t) = A_1 \cos(\omega_0 t) + A_2 \sin(\omega_0 t)$$

Using the power series method, derive the solution given above.
The following relationships may be useful:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \qquad \cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \qquad e^x = \sum_{m=0}^{\infty} \frac{x^m}{(m)!}$$

25. Assuming D is the usual differential operator $\frac{d}{dx}$,

a) does $(xD)^2 = x^2D^2$?

b) does $(x^2D^2)(xD) = (xD)(x^2D^2)$?

26. Obtain a complete set of linearly independent solutions of the following differential equation. If possible, give each solution in the form of a power series about $x = 0$.

$$y'' + xy' + 2y = 0$$

27. Obtain the first four terms in the power series solutions for the following ordinary differential equation that are valid near x=o:

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} - (x+2)y = o$$

28. Show that the following partial differential equation is separable into two ordinary differential equations by obtaining them. **DO NOT ATTEMPT TO SOLVE THESE TWO ORDINARY DIFFERENTIAL EQUATIONS, HOWEVER.**

$$[p(x)\,u_x]_x - r(x)\,u_{tt} = 0$$

29. Consider the following differential equation:

$$y'' + (y')^2 = 0$$

a) Verify that $y_1(x) = 1$ and $y_2(x) = \ln x$ each satisfy the equation.

b) Is the quantity $y = c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are constants, also a solution, in general?

30. Find the eigenvalues and eigenfunctions for the following proper Sturm-Liouville systems:

a) $y'' + \lambda^2 y = 0$ where $y(0) = 0$ and $y(L) = 0$

b) $y'' + \lambda^2 y = 0$ where $y(-L/2) = 0$ and $y(L/2) = 0$

31. Suppose the solution of some second order, linear, ordinary differential equation is:

$$y(x) = c_1 e^{-3x} + c_2 e^{7x}$$

Can you rewrite this solution in the form

$$y(x) = c_3 e^{ax} \cosh(bx) + c_4 e^{ax} \sinh(bx)$$

for appropriate values of a and b?

32. Given these facts about Bessel functions of the first kind:

$$J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$$

$$J_p'(x) = -J_{p+1}(x) + \frac{p}{x} J_p(x)$$

and $$J_0(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin\theta)\, d\theta$$

(a) deduce that $J_1(x) = \frac{2}{\pi} \int_0^{\pi/2} \sin(x \sin\theta) \sin\theta \, d\theta$

(b) integrate this result by parts to obtain:

$$J_1(x) = \frac{2x}{\pi} \int_0^{\pi/2} \cos(x \sin\theta) \cos^2\theta \, d\theta$$

(c) hence show that

$$J_2(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin\theta) \cos(2\theta) \, d\theta$$

33. Use the relationship

$$[x^n \mathrm{J}_n(x)]' = x^n \mathrm{J}_{n-1}(x) \quad ,$$

where $\mathrm{J}_n(x)$ is the Bessel function of the first kind of order n, to evaluate:

$$\int x^3 \mathrm{J}_0(x) dx \quad .$$

34. We wish to obtain the form of the solutions to Bessel's equation

$$x^2 y'' + xy' + (x^2 - \beta^2)y = 0$$

which are valid for large x. Starting with the following general second order linear differential equation with arbitrary coefficients:

$$y'' + p(x)y' + q(x)y = 0$$

consider the change of variable, $y(x) = \sigma(x)\nu(x)$, where ν is a new dependent variable and σ is a specified weight function. Show that the first derivative term can be eliminated if σ is chosen such that it satisfies

$$2\sigma' + p\sigma = 0$$

Solve for σ in the case of Bessel's equation. Hence show that Bessel's equation reduces to

$$\nu'' + [1 - \frac{(\beta^2 - 1/9)}{x^2}]\nu = 0$$

Now make a large x approximation to the coefficient of the ν - term and solve the remaining approximate differential equation. What then is the large argument approximate solution of Bessel's equation?

35. Consider the following first order differential equation:

$$\left(x^2 - 1\right)y' = xy$$

a) Solve it by the method of separable differential equations.

b) Solve it by the method for solving linear first order differential equations.

c) Solve it by the power series method assuming a power series expansion about the origin. Obtain at least the first three non zero terms in this series.

59. Obtain the solution to the boundary value problem:

$$x^2 \frac{d^2 y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^2 \qquad 1 \leq x \leq 2$$

$$y(1) = 0 \qquad y(2) = 0$$

36. The displacement, y(x) of a vibrating continuous system, fixed at both ends, obeys the following equations:

$$x^2 y'' - 2xy' + \lambda x^3 y = 0 \qquad 0 \le x \le b$$
$$y(0) = 0, \; y(L) = 0$$

a) Obtain the eigenvalues, λ_n, and eigenfunctions

b) Write out the orthogonality conditions

37. Given the following recurrence formulae involving Bessel functions:

$$\int x^m J_n(x)\, dx = x^m J_{n+1}(x) - (m-n+1) \int x^{m-1} J_{n+1}(x)\, dx$$
$$\int x^m J_n(x)\, dx = -x^m J_{n-1}(x) + (m+n-1) \int x^{m-1} J_{n-1}(x)\, dx .$$

Note that the first one will yield a closed form answer, when used iteratively, for the case where (m-n) is an odd positive integer whereas the second one will do so for the case where (m+n) is an odd positive integer. Use these results to evaluate:

$a,\ \int x^{p+1} J_p(\alpha x)\, dx = \dfrac{x^{p+1}}{\alpha} J_{p+1}(\alpha x) + c$

$b,\ \int x^3 J_0(x)\, dx$

$c,\ \int J_3(x)\, dx$

38. (a) Find the eigenvalues and eigenfunctions of the following Sturm-Liouville system:

$$y'' + \lambda^2 y = 0, \qquad y'(0) = 0, \qquad y(\pi) = 0 \; .$$

(b) Write out, explicitly, the orthogonality condition for these eigenfunctions.

(c) Suppose an arbitrary function $f(x)$ is to be expanded into a series of these eigenfunctions on the interval $0 \le x \le \pi$. Develop the expression for the Fourier weighting coefficients in this expansion.

39. Given that the Laplace transform of $J_o(x)$ (the zero order Bessel function of the first kind) is $(s^2+1)^{-1/2}$, what is the value of

$$\int_0^\infty J_o(x) dx \; ?$$

40. Solve the following differential equation using the "Power Series Method." Show at least the first three (3) terms of each independent series solution.

$$y'' + x^2 y'' + xy = 0$$

41. For the following first order differential equation

$$(x-1)\frac{dy}{dx} + y = 0$$

a. Solve it by the direct method for a first order differential equation

b. Solve by the power series method about $x_o = 0$

42. The solution of a homogeneous 5^{th} order equation with constant coefficients is:

$$y_h(x) = c_1 x e^{-2x} + c_2 e^{-2x} + c_3 x^2 + c_4 x + c_5$$

a. What is the differential equation written in terms of the linear operator L in terms of d/dx? Leave your answer in a factored operator.

b. If the equation is non-homogeneous, with

$$Ly = f(x) = 2x^5 + 2x^2 e^{-2x}$$

what would you assume for a trial particular solution by the method of undetermined coefficients. DO NOT OBTAIN THE PARTICULAR SOLUTION.

43. Obtain the homogeneous solution to the following Euler ordinary differential equation

$$x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + y = 0$$

44. Given the MacLauren expansions:

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$g(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

verify, through the first three non zero terms, that term-by-term operations on these series yield the following results:

a, $f'(x) = g(x)$

b, $\int_0^x g(y)\, dy = f(x)$

c, $f(x)\, g(x) = \frac{1}{2} f(2x)$

d, *show that* $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots \left(\textit{valid for } |x| < \frac{\pi}{2} \right)$

45. State whether each of the following functions is an even function of its real argument x, an odd function, or neither:

a) $x|x|$

b) $\sin^2(x)$

c) $\cosh(x)$

d) $J_0(x) + xJ_1(x)$ Bessel functions of first kind

e) $P_{2n+1}(x)$ Legendre polynomial, n an integer

f) $\left(\frac{1}{x} - 1\right)$

g) $\cos(x + \pi/2)$

h) $\Gamma(x)$ gamma function

i) $\sin^{-1}(x)$

j) $\ln(1+e^x) - x/2$

k) e^x

l) $\sin(nx)$ n an integer

m) $(\cos x) / x$

n) c a nonzero constant

o) e^{x^2}

46. Assuming that the functions sin(kx) and cos(kx) more naturally describe the spatial dependence of standing waves while the functions exp($\pm$ j kx) more naturally describe traveling waves, can you make a case for which Bessel functions more naturally describe standing waves as opposed to which more naturally describe traveling waves?

47. Which of the following quantities involving Bessel functions of the first kind do you think is bigger and why?

$$\int_0^\infty J_0(x)dx \qquad \text{or} \qquad \int_0^\infty J_1(x)dx$$

Note that first root of J_0 is 2.405; first root of J_1 is 3.832; first maximum of J_1 is 0.581

48. Assume for the following questions that the speed of sound in air is 340 m/s and that the density is 1.2 kg/m^3.

(a) What does it mean to say that a set of functions is linearly independent over some range of x?

(b) Is the following set linearly independent or not: (1+2x), (2+3x), (3+4x)?

49. The two functions $u_1(x)$ and $u_2(x)$ are linearly independent on some x-interval. Under what conditions are the two functions

$$f_1(x) = A_1\, u_1 + A_2\, u_2$$

and

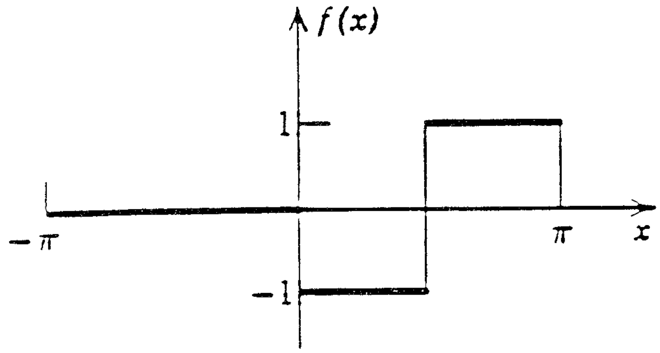
$$f_2(x) = B_1\, u_1 + B_2\, u_2$$

where A_1, A_2, B_1, B_2 are arbitrary constants, also linearly independent on that same x-interval?

50 What does it mean to say that n functions, $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ are linearly independent of one another over some range of x? Describe a test or condition or operation or something for determining whether linear dependence or linear independence obtains. Which is it? Is this test (condition, operation, ...) necessary, sufficient, or necessary and sufficient?

51. Suppose the function $f(x)$ shown in the sketch below is expanded into a Fourier series of period 2π. **DO NOT ATTEMPT TO DO THE EXPANSION.** To what value will the Fourier series converge at each of the following values of x?

(a) $x = -\pi$	(b) $x = -\pi/2$	(c) $x = 0$	(d) $x = \pi/4$
(e) $x = \pi/2$	(f) $x = 3\pi/4$	(g) $x = \pi$	(h) $x = 2\pi$



52. Given the following recurrence relationship for Bessel functions:

$$J_{n-1}(x) = J_{n-1}(x) - 2J_n'(x) \ ,$$

evaluate $\int J_3(x)dx$.

53. Consider $\displaystyle\int \frac{dx}{x \ ln \ x}$.

a) Introduce the change of variable, u = *ln* x, and proceed to obtain an answer for this indefinite integral.

b) Now try to integrate by parts where you use $u = \dfrac{1}{ln \ x}$, $dv = \dfrac{dx}{x}$.

Show that this leads to nonsense. Why doesn't this approach work?

a. What is a Wronskian? Give an example. What good is the Wronskian?

b. What do you assume for the solution of a linear differential equation with constant coefficients? Why?

c. What do you assume for the solution of an Euler (or Cauchy-Euler or equidimensional) type of differential equation? Why?

d. Is the particular solution of a linear differential equation unique? Why or why not?

e. Explain the significance of the quantity "radius of convergence."

f. What is a singularity of a linear differential equation?

g. In a few words, contrast the functional dependence upon its argument x of a Bessel function of the first kind of integer order n versus that of a <u>modified</u> Bessel function of the first kind of integer order n.

h. What distinguishes a boundary value problem from an initial value problem?

i. What does it mean to say two functions are orthogonal to one another over some interval?

j. Give an example of a periodic function of x whose Fourier series contains no cosine terms.

5b. An initial value problem is posed by the following differential equation

$$\frac{d^2y}{dt^2} + a^2y - A\sin at = 0$$

and the initial values

$$y(0) = 0 \qquad\qquad y'(0) = 0$$

(a) Give the solution of this problem in closed form for $t > 0$.

(b) The problem posed above corresponds to a physical problem where y is the displacement of a mechanical oscillator. Describe such a problem in physical terms with one or more sketches as completely as you can. Discuss the physical interpretation of the symbols a and A.

(c) Explain in physical terms why the solution derived in (a) has the behavior you found, and discuss whether there would have been any marked qualitative difference in the solution at large positive t if the term $A\sin at$ would have been instead $A\sin 2at$.

55. Evaluate the integral

$$\int_0^\infty t^{1/2} e^{-4t}\, dt$$

[In the evaluation of this integral you can use the result

$$\int_{-\infty}^{\infty} e^{-u^2}\, du = \tfrac{1}{2}\sqrt{\pi}$$

without proving it. If you can prove it and if you have time to do so, giving a proof here can do no harm.]

56. A number x is one of four roots of the quartic equation

$$(2 \times 10^{-8})x^4 - 50x^2 + 20x - 2 \times 10^{-3} = 0$$

a) Estimate the largest postive real root to 2 significant figures.

b) Estimate the smallest positive real root to 2 significant figures.

57. For what range of x does the following power series converge?

$$\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} (x+1)^{2n}$$

If a range is found, determine whether the series converges when x is at one or the other of the range's endpoints.

58 In the field of random data analysis, the auto-correlation function, $R_x(\tau)$, for stationary random data is real, is symmetric about $\tau = 0$, and the integral of its absolute value over $(-\infty, \infty)$ is finite. The auto-correlation function can be computed from the inverse Fourier transform of the auto-power spectral density function $S_x(\omega)$,

$$R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) e^{i\omega\tau}\, d\omega.$$

Prove that $S_x(\omega)$ is real-valued and an even function of ω.

57. For a homogeneous medium with a constant ambient velocity $\bar{v}_0$ one can show that the linear acoustics equations can be represented by

$$\left(\frac{\partial}{\partial t} + \bar{v}_0 \cdot \nabla\right) p + \rho c^2 \nabla \cdot \bar{v}' = 0$$

$$\rho \left[\frac{\partial \bar{v}'}{\partial t} + (\bar{v}_0 \cdot \nabla)\, \bar{v}'\right] + \nabla p = 0$$

where $\bar{v}'$ is the usual acoustic particle velocity and ρ is the ambient acoustic density.

Using these two equations, show that a corresponding wave equation for p is

$$\nabla^2 p - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + \bar{v}_0 \cdot \nabla\right)^2 p = 0 \ .$$

Although you may not need any of them, the following vector identities are provided for your convenience: ($\bar{f}$ and $\bar{g}$ are vectors; Φ and Ψ are scalars)

$\nabla(\Phi + \Psi) = \nabla\Phi + \nabla\Psi$
$\nabla(\Phi\Psi) = \Phi\nabla\Psi + \Psi\nabla\Phi$
$\nabla(\bar{f} \cdot \bar{g}) = (\bar{f} \cdot \nabla)\bar{g} + (\bar{g} \cdot \nabla)\bar{f} + \bar{f} \times (\nabla \times \bar{g}) + \bar{g} \times (\nabla \times \bar{f})$
$\nabla \cdot (\Phi\bar{f}) = \Phi(\nabla \cdot \bar{f}) + \bar{f} \cdot \nabla\Phi$
$\nabla \cdot (\bar{f} \times \bar{g}) = \bar{g} \cdot (\nabla \times \bar{f}) - \bar{f} \cdot (\nabla \times \bar{g})$
$\nabla \times (\Phi\bar{f}) = \Phi(\nabla \times \bar{f}) + (\nabla\Phi) \times \bar{f}$
$\nabla \times (\bar{f} \times \bar{g}) = \bar{f}(\nabla \cdot \bar{g}) - \bar{g}(\nabla \cdot \bar{f}) + (\bar{g} \cdot \nabla)\bar{f} - (\bar{f} \cdot \nabla)\bar{g}$
$\nabla \times (\nabla \times \bar{f}) = \nabla(\nabla \cdot \bar{f}) - \nabla^2\bar{f}$
$\nabla \times (\nabla\Phi) = 0$
$\nabla \cdot (\nabla \times \bar{f}) = 0$
$\nabla^2(\Phi\Psi) = \Phi\nabla^2\Psi + 2(\nabla\Phi) \cdot (\nabla\Psi) + \Psi\nabla^2\Phi$

60. The velocity potential $\phi(x, y)$ is related to u and v (as in 2. above) by:

$$\frac{\partial\phi}{\partial x} = u \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = v.$$

Assume the z-component of velocity (w) is zero. The condition for irrotationality is $\vec{\nabla} \times \vec{V} = 0$, where $\vec{V}$ has components (u, v). Prove that the velocity given in problem 2. satisfies irrotationality. Also, derive an equation for $\phi(x, y)$.

61. Consider 2-D, steady, irrotational fluid motion. The stream function maps the trajectories of fluid particles in motion. It is orthogonal to the velocity potential. Given the following two equations for the stream function $\Psi(x, y)$ that relate $\Psi(x, y)$ to the x- and y-components of velocity, respectively:

$$\frac{\partial \Psi}{\partial y} = u \qquad (x - component)$$

$$-\frac{\partial \Psi}{\partial x} = v \qquad (y - component),$$

find Ψ for

$$u = ax^2 - ay^2$$

$$v = -2axy,$$

where a is a constant.